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Mathematical Reviews

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NUMBER THEORY

Goodstein, R. L. Recurring digits in irrational decimals. Math. Gaz. 25, 273–278 (1941). [MF 6270]

Rosser, Barkley. A note on the linear Diophantine equation. Amer. Math. Monthly 48, 662–666 (1941). [MF 6025]

This paper presents an alternative to the methods presented in a paper by Lehmer [Amer. Math. Monthly 48, 240–246 (1941); these Rev. 2, 247]. It leads to a general solution with smaller coefficients. *R. D. James.*

Barnett, I. A. and Mendel, C. W. Pythagorean points lying in a plane. Amer. Math. Monthly 48, 610–616 (1941). [MF 5895]

The authors study the system of Diophantine equations $x^2+y^2=z^2$, $Ax+By+Cz=D$, with A, B, C relatively prime, that is, they determine the lattice points on a certain conic section in space. By using the solutions $x=\frac{1}{2}\rho(u^2-v^2)$, $y=\rho uv$, $z=\frac{1}{2}\rho(u^2+v^2)$ of the first equation, the second equation becomes $(C+A)u^2+2Buv+(C-A)v^2=2D/\rho$. Hence $2D/\rho$ has to be represented by this binary quadratic form in u, v; the discriminant of this form is $\Delta=A^2+B^2-C^2$. In addition certain conditions of parity have to be satisfied. When $D=0$ (degenerate conic) the system is homogeneous and solutions exist if and only if Δ is a perfect square. When $D\neq 0$ there are three cases: (1) $\Delta<0$ (ellipse), the number of solutions is finite (possibly 0); (2) $\Delta=0$ (parabola), there are infinitely many solutions; (3) $\Delta>0$ (hyperbola), there are infinitely many solutions or none at all. These results are obtained without explicit use of the theory of quadratic forms, although properties of the Pellian equation are used in the discussion of case 3.

H. W. Brinkmann (Swarthmore, Pa.).

Podsipanin, W. D. Über eine unbestimmte Gleichung wo $\sigma=1, 2, 4, 8, 16$ ist. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 5, 305–324 (1941). (Russian. German summary) [MF 5838]

The exact English translation of the title is "On a certain indeterminate equation."

The equation considered is (1) $ax^2-by^4=\sigma$, $\sigma=1, 2, 4, 8, 16$, or rather (2) $px^2-my^4=16$ with $p=16a/\sigma$, $m=16b/\sigma$. The author shows that for a given (p, m) only a finite number of quartic fields K need be considered. In each of these fields the author finds a fundamental unit ϵ. For $K\neq R((-m)^{\frac{1}{4}})$ one constructs the number $\mu=(\epsilon(-m)^{\frac{1}{4}})^{\frac{1}{2}}$. If this number satisfies a quadratic equation of the form $\mu^2-w(-m)^{\frac{1}{4}}\mu-(-m)^{\frac{1}{2}}=0$, then $y=w$ is a solution of (2) whose corresponding x can then be found. A finite number of other solutions (usually none) can be found by considering the quadratic equations satisfied by $\mu^{2n+1}(-m)^{-n}$, where n is an integer such that $\epsilon^{2n+1}=\epsilon$. If μ satisfies no such quadratic equation then (2) is impossible in integers. In case $K=R((-m)^{\frac{1}{4}})$ the equation reduces to $a'x^4-b'y^4=\sigma$, $\sigma=1, 2, 4, 8$, already considered by Faddeev [Uchenye Zapiski Leningrad. Gosudarstvennogo Pedagogicheskogo Inst. im. Hertzena 28, 141–145 (1939)]. There are a number of special exceptional equations which are all disposed of, except the equation $17x^2-y^4=16$, which may or may not have solutions other than $x=\pm1$, $y=\pm1$. The casual reader should be warned that the a, b used in the body of the paper are not the coefficients of (1).

D. H. Lehmer (Berkeley, Calif.).

Venkata Rama Ayyar, M. A note on three allied problems of Ramanujan. J. Mysore Univ. Sect. B. 1, 109–114 (1941). [MF 5137]

Ramanujan has given special sets of integral solutions of the Diophantine equations (1) $x^3+y^3+z^3=u^3$, (2) $x^3+y^3+z^3=1$, (3) $x^3+y^3+z^3=u^6$. He found for x, y, z and u quadratic expressions in two variables which satisfy (1) identically. Equation (2) has been discussed by Mitra [Amer. Monthly 32, 506–509 (1925)] and by Mahler [J. London Math. Soc. 11, 136–138 (1936)]. The author gives an interesting method modeled along the lines of Mitra yielding solutions of $x^3+y^3=k^3(z^3+w^3)$ for $k=2, 3, 4$.

I. A. Barnett (Cincinnati, Ohio).

Griffiths, L. W. Universal functions of extended polygonal numbers. Amer. J. Math. 63, 726–728 (1941). [MF 5620]

This paper supplements the results of a previous paper [Amer. J. Math. 55, 102–110 (1933)] on the problem of determining which linear functions

$$a_1t_1+a_2t_2+\cdots+a_nt_n,\quad a_1+a_2+\cdots+a_n\leqq m;\ m>5,$$

represent every number when the t's range independently over all "extended" polygonal numbers $-x+m(x^2+x)/2$ $(x=-1, 0, 1, 2, \cdots)$ of order m. Certain necessary conditions were found in the previous paper and in some cases these were found sufficient. It is now found that the possibly exceptional cases of the previous paper are also sufficient. The methods of proof, which appear to be complicated, are illustrated by considering the representation of all numbers lying between $151m-22$ and $153m-23$. *D. H. Lehmer.*

Aigner, Alexander. Die Zerlegung einer arithmetischen Reihe in summengleiche Stücke. Deutsche Math. 6, 77–89 (1941). [MF 5806]

The paper contains an interesting examination of questions centered around the following problem: consider an arithmetical progression with the νth term $a_\nu=a+(\nu-1)d$, $d\geqq 1$, $a\gtreqless 0$, $(a, d)=1$, all numbers rational integers. The progression is called zweiteilig when it has the property that for some $k>1$ and $n>k$ we have $a_1+\cdots+a_k=a_{k+1}+\cdots+a_n$. Letting $\sum_\nu$ be the sum of the first ν terms, we have $\sum_n=2\sum_k$, from which $(d-2a)(n-2k)=d(n^2-2k^2)$. This shows that the problem has no solution for $d=2a$ or $n=2k$. If n, k have assigned values, there is always one solution, from $a:d=(n^2-2k^2-2):(4-2n)$, except

when $n=2k$. Assuming next a, d given, and solving the quadratic equation for n, the discriminant is found to be $D=2(2a-d+2dk)^2-(2a-d)^2$. With $D=x^2$ (since n must be an integer), $2a-d+2dk=y$, we are led to the Diophantine equation $x^2-2y^2=-(2a-d)^2$; $2n=(x-2a+d)/d$, $2k=(y-2a+d)/d$; $2a_n=x-d$, $2a_k=y-d$. The investigation is thus reduced to the examination of a Pell equation. It is proved that to every $d(>0)$ and every $a>d/2$ there exist an infinite number of zweiteilig progressions (solutions). The case $a\leqq d/2$ leads to arithmetical difficulties. It is found: (I) $a<-3d$: infinite number of solutions, or a finite number, or no solutions; (II) $-3d\leqq a\leqq d/2$: infinite number of solutions, or no solution. These possibilities are discussed in detail. Examples: impossible $(a, d)=(1, 4)$, $(-1, 4)$, $(1, 7)$, $(3, 7)$; possible: $1+4+\cdots+13$; $3+7+\cdots+27$; $2+3+\cdots+7$; $2+3+\cdots+43$.

A second part deals with the size of the solution n, k, or a_k, a_n in the cases when an infinite number of solutions exist for a given a, d. Here $\lim k/n=\lim a_k/a_n=1/\sqrt{2}$, independently of a, d. This follows from $n=(x-2a+d)/2d$, $k=(y-2a+d)/2d$; $2a_n=x-d$, $2a_k=y-d$, $x^2-y^2=-(2a-d)^2$, and the known behavior of solutions of Pell's equation. Part 3 considers the set of natural numbers, the progression $1+2+3+\cdots$. Now $x^2-2y^2=-1$, $2n=x-1$, $2k=y-1$, and the solutions are given by $(x+y\sqrt{2})=(1+\sqrt{2})^{2h+1}$, $h>0$. The first six solutions are $(x, y; n, k)=(7, 5; 3, 2)$; $(41, 29; 20, 14)$; $(239, 169; 119, 84)$; $(1393, 985; 696, 492)$; $(8119, 5741; 4059, 2870)$; $(47321, 33461; 23660, 16730)$. For the second, for example, $41^2-2\cdot 29^2=-1$, $n=(41-1)/2=20$, $k=(29-1)/2=14: 1+2+\cdots+14=15+\cdots+20$. Part 4 proves the impossibility of $a_1+\cdots+a_k=a_{k+1}+\cdots+a_n=a_{n+1}+\cdots+a_m$ except for the trivial case $d=0$ (dreiteilig impossible).

Finally, general arithmetical progressions $a, a+d, a+2d, \cdots$, with arbitrary real or complex a, d, are considered. It is easily shown that every zweiteilig arithmetical progression of real or complex numbers is a zweiteilig progression of real integers, multiplied by a constant factor. Any arithmetical progression $(d\neq 0)$ of real or complex numbers is at most zweiteilig (never dreiteilig or μ-teilig, $\mu>2$).

A. J. Kempner (Boulder, Colo.).

Wahlgren, Agne. Numbers which can be represented as a sum of four cubes. Mat. Tidsskr. B. **1941**, 33–41 (1941). (Swedish) [MF 6173]

Niven, Ivan. Sums of fourth powers of Gaussian integers. Bull. Amer. Math. Soc. **47**, 923–926 (1941). [MF 5939]

The author proves that a Gaussian complex integer is expressible as a sum of fourth powers of such integers if and only if the given integer is of the form $a+24bi$ (a, b rational integers). It is also shown that at most 18 such fourth powers are necessary. The methods used are elementary and in particular the author uses his theorem that any Gaussian integer of the form $a+2bi$ can be expressed as the sum of two squares [Trans. Amer. Math. Soc. **48**, 405–417 (1940), in particular, p. 410; these Rev. **2**, 147].

H. W. Brinkmann (Swarthmore, Pa.).

Davenport, H. On sums of positive integral k-th powers. Amer. J. Math. **64**, 189–198 (1942). [MF 5997]

In a previous paper with the same title [Proc. Roy. Soc. London. Ser. A. **170**, 293–299 (1939)] the author established certain lower bounds for $N_s^{(k)}(n)$, the number of integers less than n that are representable as a sum of s positive kth powers. In the present paper more precise results of the same type are proved. A sample result is the following: If $N_s^{(k)}(n)>n^\alpha$ for all large n, where $1/k<\alpha<1$, then $N_{s+1}^{(k)}(n)>n^{\beta-\epsilon}$ for $n>n_0(\epsilon)$, where

$$\beta=\max_{h\leq k-2}\frac{1}{k}\left\{1+\frac{(2^h-1)(k-1)+h+1}{2^h-1+\alpha}\alpha\right\}.$$

The proof depends on an estimate of the number of solutions of $x^k+u_i=y^k+u_j$ subject to the conditions $P<x$, $y<2P$, $u_i, u_j<P^{k\lambda}$, $\lambda=1-(1-\delta)/k$, $0<\delta<1$. More precise results are obtained for the special cases $k=5$, $k=6$.

R. D. James (Madison, Wis.).

Davenport, H. On Waring's problem for fifth and sixth powers. Amer. J. Math. **64**, 199–207 (1942). [MF 5998]

Let $G(k)$ denote the least value of s for which every sufficiently large positive integer is representable as the sum of s positive integral kth powers. Using the lower bounds for $N_s^{(5)}(n)$ and $N_s^{(6)}(n)$ obtained in the paper reviewed above, the author shows that $G(5)\leqq 23$, $G(6)\leqq 36$, improvements on $G(5)\leqq 28$, $G(6)\leqq 42$. The method of proof is similar to that given for fourth powers [Ann. of Math. (2) **40**, 731–747 (1939); these Rev. **1**, 42].

R. D. James (Madison, Wis.).

Davenport, H. Note on sums of fourth powers. J. London Math. Soc. **16**, 3–4 (1941). [MF 5897]

This note contains an elementary proof of the result that every integer n is representable as $n=\pm x_1^4\pm x_2^4\pm\cdots\pm x_{11}^4$ with suitable signs depending on n. *R. D. James.*

Hunter, W. The representation of numbers by sums of fourth powers. J. London Math. Soc. **16**, 177–179 (1941). [MF 6016]

H. Davenport has recently shown [cf. the preceding review] that, if $V(4)$ is the least value of s for which every integer n is representable in the form $n=\pm x_1^4\pm x_2^4\pm\cdots\pm x_s^4$ (the x's all being integers and a suitable choice of signs being made for each n), then $V(4)\leqq 11$. The author proves that $V(4)=9$ or 10 and that n can be represented in an infinity of ways as such a sum with $s=10$. This is done by the use of identities. *B. W. Jones* (Ithaca, N. Y.).

Sambasiva Rao, K. On Waring's problem for smaller powers. J. Indian Math. Soc. (N.S.) **5**, 117–121 (1941). [MF 6129]

This note contains proofs that $G(5)\leqq 24$, $G(6)\leqq 36$, $G(7)\leqq 52$ using the method of Davenport.

R. D. James (Madison, Wis.).

Narasimhamurti, V. On Waring's problem for 8th, 9th, and 10th powers. J. Indian Math. Soc. (N.S.) **5**, 122 (1941). [MF 6130]

This note describes a method for establishing the results $G(8)\leqq 73$, $G(9)\leqq 99$, $G(10)\leqq 122$. *R. D. James.*

Chakrabarti, M. C. On the limit points of a function connected with the three-square problem. Bull. Calcutta Math. Soc. **32**, 1–6 (1940). [MF 6151]

Let $c(x)$ denote the number of integers not greater than x which can be represented as a sum of at most three integral squares; let $\psi(x)=(c(x)-5x/6)/\log x$. It is known that $\psi(x)=O(1)$. In this paper it is proved that (a) $\underline{\lim}\,\psi(x)=0$; (b) $\overline{\lim}\,\psi(x)=1/\log 8$; (c) every point of the interval $(0, 1/\log 8)$ is a limit point of $\psi(x)$. The method of proof

is to find a formula for $c(x)$ based on the representation of x in the scale of 2. *R. D. James* (Madison, Wis.).

Maass, Hans. Über die Darstellung total positiver Zahlen des Körpers $R(\sqrt{5})$ als Summe von drei Quadraten. Abh. Math. Sem. Hansischen Univ. 14, 185–191 (1941). [MF 5443]

It is known that the number of representations of a positive rational integer n as a sum of three integral squares is closely related to the number of classes of ideals in the quadratic field $R(\sqrt{-n})$. The author finds an analogous relationship connecting the number $a(\mu)$ of representations of a totally positive integer μ in the field $R(\sqrt{5})$ as a sum of three integral squares with the number of classes of ideals in the biquadratic field $R(\sqrt{5}, \sqrt{-\mu})$. He proves that there exists in $R(\sqrt{5})$ only one class of ternary quadratic forms with integral coefficients and determinant 1; consequently the theta series and the analytic genus invariant $\phi(\tau)$ belonging to the quadratic form $x_1^2+x_2^2+x_3^2$ are identical, and the theorem follows from the expression of $\phi(\tau)$ as a power series. Moreover, it is proved that $a(\mu)$ is always positive; hence every totally positive integer in $R(\sqrt{5})$ can be written as the sum of three integral squares.

C. L. Siegel (Princeton, N. J.).

Siegel, Carl Ludwig. Equivalence of quadratic forms. Amer. J. Math. 63, 658–680 (1941). [MF 4691]

If R, R_p, R_∞ are the fields of rational, p-adic, real numbers, respectively, and J and J_p the rings of integral numbers and of p-adic integers, respectively, the main object of this paper can be stated in the two theorems: (I) If two quadratic forms with coefficients of R are equivalent in all R_p and in R_∞ then they are also equivalent in R. (II) If two quadratic forms with coefficients of J are equivalent in all J_p and in R_∞ then they are semiequivalent. Semiequivalence of two quadratic forms means that the transformation matrix can be chosen so as to have elements with denominators prime to a preassigned integer q. Theorem (I) was found by Minkowski and proved in detail first by Hasse. Theorem (II) was proved for three variables by Smith, stated in general by Minkowski, but not completely proved before this paper.

It is easily seen that only nondegenerate forms, and for (I) in particular only diagonal forms with integral coefficients, need to be considered. Instead of Dirichlet's theorem on the prime numbers in arithmetical progressions, which had been used in previous proofs, the author employs here the properties of theta-functions, which he considers the appropriate tools for the investigation of quadratic forms. To the indefinite form $S=\sum_{k=1}^m a_k x_k^2$ with integral coefficients Siegel associates the form $P=\sum_{k=1}^m |a_k| x_k^2$ and enumerates the solutions of $S=0$ by $A(\epsilon)=\sum_{S=0}\exp(-\pi\epsilon P)$, for an arbitrary $\epsilon>0$ and where $x_1, \cdots, x_m$ run over all solutions of $S=0$. Introducing

$$f(u)=\sum_x \exp(-\pi\epsilon P+2\pi i u S),$$

where the sum is extended over all integral values $x_1, \cdots, x_m$, we have clearly that $A(\epsilon)=\int_0^1 f(u)du$. On this equation Siegel applies the Hardy-Littlewood method of Farey arcs and studies in particular the emerging singular series in order to prove: (III) For $m>4$ we have $\lim_{\epsilon\to 0}\epsilon^{(m/2)-1}A(\epsilon)\to\omega\sigma>0$ with explicitly known constants ω and σ depending on the quadratic form S. (IV) If for $m=4$ the indefinite form S represents zero in all R_p and if its determinant is a square then $(\epsilon/|\log\epsilon|)A(\epsilon)$ tends to a positive limit with $\epsilon\to 0$. These theorems are quantitative refinements of theorems due to Legendre, H. Hasse and A. Mayer.

Theorem (I) requires an induction in the number of variables which is based on a theorem of E. Witt, here specialized to the lemma: If S and T are two quadratic forms in the variables $x_1, \cdots, x_m$ and with coefficients in one of the fields R, R_p, R_∞ and if ax_0^2+S and ax_0^2+T are equivalent in the field under consideration, then S and T are also equivalent in that field. Now in Theorem (I) two quadratic forms

$$S=\sum_{k=1}^m a_k x_k^2, \quad T=\sum_{k=1}^m b_k y_k^2$$

with integral coefficients are given which are equivalent in all R_p and in R_∞. Then the author considers the quadratic form $V=S-T$ in $2m$ variables $x_1, \cdots, x_m, y_1, \cdots, y_m$. On account of the equivalence of S and T in R_∞ the form V is indefinite and on account of the equivalence in all R_p the form V represents zero in all R_p. This makes it possible to apply Theorems (III) and (IV) to the proof of (I). Theorem (II) is derived from (I) by means of some additional algebraic considerations. *H. Rademacher* (Philadelphia, Pa.).

Hua, Loo-keng. A note on the class number of ternary quadratic forms. J. London Math. Soc. 16, 82–83 (1941). [MF 5123]

To prove that the number of classes of positive ternary quadratic forms of determinant d tends to infinity with d, Hua notes that the forms $x^2+f(y, z)$ are non-equivalent, where f ranges over the $h(d)$ binary forms of determinant d, and that $h(d)\to\infty$. The result follows, we may observe, somewhat more simply from the Eisenstein-Smith formula for the weight of a ternary genus. *G. Pall.*

Witt, Ernst. Eine Identität zwischen Modulformen zweiten Grades. Abh. Math. Sem. Hansischen Univ. 14, 323–337 (1941). [MF 5449]

Let $(u_{k1}, \cdots, u_{km})$ $(k=1, \cdots, m)$ be a basis of the lattice of all vectors $(u_1, \cdots, u_m)$ satisfying the conditions $u_k\equiv 0 \pmod{\frac{1}{2}}$, $u_k\equiv u_l \pmod 1$, $\sum_{k=1}^m u_k\equiv 0 \pmod 2$, where m denotes a multiple of 8. Then

$$Q_m(x)=\sum_{l=1}^m (u_{1l}x_1+\cdots+u_{ml}x_m)^2$$

is an even positive quadratic form of determinant 1. For $m=16$, the genus of Q_m contains exactly two classes, namely, the class of $Q_{16}(x)$ and the class of $Q_8(x)+Q_8(y)$. The two non-equivalent quadratic forms $Q_{16}(x)$ and $Q_8(x)+Q_8(y)$ represent every binary quadratic form the same number of times. The proof uses the analytic theory of quadratic forms and the theory of modular forms of degree 2. The result can be expressed by the identity $(\sum|\mathfrak{A}\mathfrak{Z}+\mathfrak{B}|^{-4})^2=\sum|\mathfrak{A}\mathfrak{Z}+\mathfrak{B}|^{-8}$, where $\mathfrak{Z}$ is a symmetric matrix with two rows and positive imaginary part, and $\mathfrak{A}, \mathfrak{B}$ run over a complete system of non-associate coprime symmetric pairs of matrices with two rows. *C. L. Siegel.*

Canaday, E. F. The sum of the divisors of a polynomial. Duke Math. J. 8, 721–737 (1941). [MF 5955]

Let A be one of the 2^n polynomials $A(x)=x^n+a_1x^{n-1}+\cdots+a_n$ with $a_k=0, 1$. Then another such polynomial is $S(A)=B$ defined as the sum of the divisors of A, the coefficients in the sum being reduced modulo 2. For n even and in some other cases, the operation S is proved to leave the sum $a_1+a_2+\cdots+a_{n-1}$ invariant (mod 2). If $B=S(A_i)$ for w

distinct polynomials A_j, then B is said to be of weight w and its divisibility by x and $x+1$ is discussed for w positive. If $S(A_j)=A_{j+1}$ with the A_j distinct for $j=1, 2, \cdots, m$ but with $A_{m+1}=A_1$, then the sequence $(A_1, A_2, \cdots, A_m)$ is said to form an m-ring. The construction of 1-rings, 2-rings and other special rings is given. Also reference is made to the tables by the author, available at Duke University library, regarding the 2046 polynomials $A(x)$ of degrees 1 to 10 and their properties. *M. Marden* (Milwaukee, Wis.).

Ivanov, V. On properties of the coefficients of the irreducible equation for the partition of the circle. Uspekhi Matem. Nauk 9, 313–317 (1941). (Russian) [MF 4996]

Let X_m denote the irreducible factor of the polynomial x^m-1 whose zeros are the primitive mth roots of unity. The present note proves the following properties of X_m (solution of a problem proposed by N. G. Tschebotareff). (i) If $m=pq$ is the product of two distinct primes, then the coefficients of various powers of x in X_m have the values $-1, 0, 1$ only. (ii) If m is a product of three distinct primes, $m=pqr$ $(p<q<r,\ p+q>r)$, then in X_m the coefficient of x^r is -2. (iii) If m has a sufficiently large number of distinct prime factors, then there exist some coefficients of X_m which exceed in absolute value any preassigned positive number. *J. A. Shohat* (Philadelphia, Pa.).

Chung, Kai-Lai. A generalization of an inequality in the elementary theory of numbers. J. Reine Angew. Math. 183, 193–196 (1941). [MF 5723]

Theorem: Given n sets of b numbers each (all the numbers considered here being positive integers):

$$\mathfrak{S}_1 \equiv \{m_1^{(1)}, \cdots, m_b^{(1)}\}, \cdots, \mathfrak{S}_n \equiv \{m_1^{(n)}, \cdots, m_b^{(n)}\}.$$

Any set $\{r_1, \cdots, r_b\}$ is said to be divisible by $\mathfrak{S}_\nu$ if, and only if, $m_j^{(\nu)} | r_j$ $(j=1, 2, \cdots, b)$, and is said to be prime to $\mathfrak{S}_\nu$ if, and only if, $(m_j^{(\nu)}, r_j)=1$ $(j=1, 2, \cdots, b)$. By ${}_bA_n$ we denote the number of sets $\{r_1, \cdots, r_b\}$, where $1 \leqq r_j \leqq \prod_{\nu=1}^n m_j^{(\nu)}$ $(j=1, 2, \cdots, b)$, which are not divisible by any of the given sets $\mathfrak{S}_\nu$ $(\nu=1, 2, \cdots, n)$, two sets being already regarded as different if they consist of the same numbers in different order. Then for any system of sets $\mathfrak{S}_\nu$ the following inequality holds:

$$\text{(II)} \qquad {}_bA_n \geqq \prod_{\nu=1}^n P_\nu(1-1/P_\nu),$$

where $P_\nu = m_1^{(\nu)} m_2^{(\nu)} \cdots m_b^{(\nu)}$ denotes the product of all numbers in $\mathfrak{S}_\nu$. Moreover, equality holds only in the case of pairwise prime sets $\mathfrak{S}_\nu$, or when at least one of the $\mathfrak{S}_\nu$ consists of 1's solely. This theorem is proved by induction on n. It represents a generalization of a theorem by H. Rohrbach [J. Reine Angew. Math. 177, 193–196 (1937)], which is, in turn, a generalization of Euler's

$$\phi(m) = m \cdot \Pi(1-1/p_\nu).$$

In a letter to the editors, the author claims the following further extension and its proof: In an algebraic number field, n ideals $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_n$ may be given. Let A_n denote the number of integers in a complete representative residue system mod $(\mathfrak{a}_1 \cdot \mathfrak{a}_2 \cdots \mathfrak{a}_n)$ which are not divisible by any $\mathfrak{a}_i$. Then

$$A_n \geqq N(\mathfrak{a}_1 \cdots \mathfrak{a}_n) \prod_{i=1}^n (1-1/N(\mathfrak{a}_i)),$$

where equality holds only in the case of pairwise prime ideals. *A. J. Kempner* (Boulder, Colo.).

Skrylew, W. Endliche Kettenbrüche, welche vermöge quadratischer Irrationalitäten gebildet sind. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 17, 145–165 (1940). (Russian. German summary) [MF 4756]

The author calls a quadratic field "Euclidean" if Euclid's algorithm is valid in it (in such a field a simple fraction is expressible as a finite continued fraction), and, writing $\bar{\alpha}$ for the number conjugate to α, calls a finite continued fraction (with elements in a Euclidean field) pseudo-symmetrical if it has the form $(\alpha, \beta, \gamma, \cdots, \bar{\gamma}, \bar{\beta}, \bar{\alpha})$. Necessary and sufficient conditions are given that a simple fraction be expressible in pseudo-symmetrical form. An algorithm is also given by means of which all pseudo-symmetrical fractions equal to a given simple fraction may be determined. *D. C. Spencer* (Cambridge, Mass.).

Brandt, H. Über die Zerlegungsgesetze der rationalen Zahlen in Quaternionen-Körpern. Math. Ann. 117, 758–763 (1941). [MF 5655]

Let G be a primitive, positive definite, quaternary quadratic form of discriminant $\Delta=d^2$, d integral. The author defines ideal classes of substitutions S, of determinant m^2, which transform G into mF, where F is a primitive form of discriminant Δ. This classification of substitutions S defines a representative system of forms F_{ik} $(i, k=1, \cdots, h)$ having the compositions $F_{ij}F_{jk}=F_{ik}$. Let $\lambda(m)$ be the matrix $(\lambda_{ik}(m))$, where $\lambda_{ik}(m)=F_{ik}(m)/F_{kk}(1)$, and $F_{ik}(m)$ is the number of representations of m by F_{ik}. The author proves that $\lambda(m_1)\lambda(m_2)=\lambda(m_2)\lambda(m_1)$ for any integers m_1 and m_2, a lemma employed by Hecke [Danske Vid. Selsk. Math.-Fys. Medd. 17, no. 12 (1940); these Rev. 2, 251] to prove the existence of an Euler product for the system of ζ-functions of the forms F_{ik}. *R. Hull* (Vancouver, B. C.).

Kiokemeister, Fred. The parastrophic criterion for the factorization of primes. Trans. Amer. Math. Soc. 50, 140–159 (1941). [MF 4872]

Let H be an extension of degree n of the field of rational numbers R and K a subdomain of H with the basis $e_1, \cdots, e_n$ over the domain of rational integers G. If p is a rational prime, the difference ring $K/\mathfrak{A}$, $\mathfrak{A}=Kp$, is an algebra over the field of residues mod p. For any algebra with the basis $e_1, \cdots, e_n$ and the multiplication table $e_ie_j=\sum \gamma_{ijk}e_k$, the matrix $Q(\xi)=(\sum \gamma_{ijk}\xi_k)$ is the parastrophic matrix and its determinant $\pi(\xi)$ is the parastrophic form of the algebra. The main result of the present paper is the establishment of a connection between the factorization of $\pi(\xi)$ into irreducible factors mod p and the factorization of $\mathfrak{A}$ in K. A refined analysis of commutative Frobenius algebras [Nakayama, Ann. of Math. (2) 40, 611–633 (1939); 42, 1–21 (1941); cf. these Rev. 1, 3; 2, 344] over a perfect field is required for the proof. We note the following results: A necessary and sufficient condition that a commutative primary algebra A over a perfect field be a Frobenius algebra is that A have only one minimal ideal. If F is an algebraic extension field of the perfect field P the parastrophic form of F over P is irreducible. It may be remarked that the assumption made throughout the discussion of commutative algebras that the underlying field be perfect may be replaced by the weaker condition that the algebra A/N, N the radical, be separable in the sense that it remains semisimple under arbitrary extension of the field. *N. Jacobson* (Chapel Hill, N. C.).

Specht, Wilhelm. Primteiler von Zahlenfolgen. II. Deutsche Math. 6, 89–96 (1941). [MF 5807]

[The first part appeared in Deutsche Math. 3, 689–697 (1939).] Let $a_1<a_2<\cdots<a_n<\cdots$ be an infinite sequence of integers. Denote by $\psi(a_n)$ the number of primes p such that $p|a_i$ for some $i\leqq n$. The author gives several estimations for $\psi(a_n)$. Among others he proves the following result: Let

$$\varliminf_{n\to\infty}\frac{\operatorname{loglog} a_n}{\log n}=0.$$

Then, for sufficiently large n,

$$\psi(a_n)>\frac{\log n}{\operatorname{loglog} a_n}.$$

P. Erdös (Philadelphia, Pa.).

Erdös, P. On some asymptotic formulas in the theory of the "factorisatio numerorum." Ann. of Math. (2) 42, 989–993 (1941). [MF 5526]

Let $1<a_1\leqq a_2\leqq\cdots$ be a sequence of integers. Denote by $f(n)$ the number of representations of n as a product of the a's, where two representations are considered equal only if they contain the same factors in the same order. The author assumes $\sum a_n^{-1-\epsilon}$ convergent for $\epsilon>0$ and that not all a's are powers of a_1. He shows by elementary considerations that

$$F(n)=\sum_{r=1}^{n}f(r)=Cn^{\rho}[1+o(1)],$$

where $\sum a_n^{-\rho}=1$. For the case $a_n=n+1$, he states the existence of constants c_1 and c_2, $0<c_2<c_1<1$, such that $f(n)>n^{\rho}\exp[-(\log n)^{c_1}]$ for infinitely many n and $f(n)<n^{\rho}\exp[-(\log n)^{c_2}]$ for all large n. These results extend previous results of L. Kalmár [Acta Litt. Sci. Szeged 5, 95–107 (1930)] and the reviewer [Acta Arith. 2, 134–146 (1937), where the result $f(n)<n^{\rho-\epsilon}$ for infinitely many n is proved]. *E. Hille* (Palo Alto, Calif.).

Erdös, P. On the asymptotic density of the sum of two sequences. Ann. of Math. (2) 43, 65–68 (1942). [MF 6061]

If $a_1<a_2<\cdots$ is an infinite sequence of integers and $f(n)$ the number of a's which do not exceed n then $\alpha=\liminf f(n)/n$ is called the asymptotic density of $\{a_n\}$. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of integers with asymptotic densities α and β, where $\alpha+\beta\leqq1$, $\beta\leqq\alpha$ and $b_1=1$. The author proves that the asymptotic density of the double sequence $\{a_i+b_j\}$ is not less than $\alpha+\beta/2$. *M. Kac* (Ithaca, N. Y.).

Vinogradoff, I. M. Two theorems relating to the theory of distribution of prime numbers. C. R. (Doklady) Acad. Sci. URSS (N.S.) 30, 287–288 (1941). [MF 4380]

Estimates are stated for

$$S=\sum_{N_1<p\leqq N_2}\exp(2\pi i f(p)),$$

where f need not be an integral polynomial. The case $f(x)=kx^{\alpha}$, $0<\alpha<1$, was treated earlier [Rec. Math. [Mat. Sbornik] N.S. 7 (49), 365–372 (1940); these Rev. 2, 40]. Let $N\geqq8$, $\frac{1}{2}N<N_1<N_2\leqq N$. Theorem 1: Given $0<\eta\leqq1/6$, $A>0$ and f with property H_1 (namely, $f''(z)$ and $f'''(z)$ continuous in the interval $\frac{1}{2}N\leqq z\leqq N$, and satisfying $A^{-1}\ll|f''(z)|\ll A^{-1}$, $A^{-1}\ll|zf'''(z)+2f''(z)|\ll A^{-1}$); then $S\ll N^{1+\eta}(A^{-1}N+AN^{-2})^{1/4}$. Theorem 1*: With the same hypotheses, $\epsilon<2\eta$ and $0<\sigma\leqq1$, then the number of primes $p\leqq N$ such that $0\leqq f(p)-[f(p)]<\sigma$ is $\sigma\pi(N)+O(N^{1+\epsilon}\Delta)$, $\Delta=(A^{-1}N+AN^{-2})^{1/5}$. Theorem 2: Given $0<\epsilon<0.01$, $k\geqq12$, $n=[2k+2]$ and f with property H_2 (namely, in the interval $\frac{1}{2}N\leqq z\leqq N$ and for $s=[k+2], [k+3], \cdots, n$, $f^{(s)}(z)$ continuous and satisfying $N^{k-s-0.01}\ll|f^{(s)}(z)|\ll N^{k-s+\epsilon}$, $N^{k-s-0.01}\ll|sf^{(s)}(z)+zf^{(s+1)}(z)|\ll N^{k-s+\epsilon}$); then $S\ll N^{1-0.133\rho}$, $\rho=(n^3\log 4n)^{-1}$. *G. Pall* (Montreal, Que.).

Vinogradoff, I. M. Some general property of distribution of products of prime numbers. C. R. (Doklady) Acad. Sci. URSS (N.S.) 30, 681–682 (1941). [MF 4469]

Results are announced pertaining to the distribution not of primes p as in earlier articles, but of numbers P running over one of the ranges: (a) quadratfrei numbers, (b) numbers with a fixed number $h(>1)$ of prime factors, (c) numbers with an even (or odd) number of prime factors. Theorem 1: $N>2$, $\alpha=a/q+\theta/q^2$, $(a,q)=1$, $0<q<N$; then

$$\sum_{P\leqq N}e^{2\pi i\alpha P}\ll N^{\epsilon}K^{\frac{1}{2}},$$

where $K=N^{-\frac{1}{2}}+q/N+k/q+k^4/q^2$. Theorems 2, 3 and 6 are immediate extensions with P in place of p, of Theorems 1, 2 and 1* of the preceding review. Theorem 4: If $0<\sigma\leqq1$, $N\geqq8$, q an integer, $0<q<N$, then the number of values of P, not exceeding N, which fall into intervals of the form $cq<P\leqq(c+\sigma)q$ for integers c, is given by $\sigma T(1)+O(N^{1+\epsilon}\Delta)$, $\Delta^2=N^{-\frac{1}{2}}+q/N+1/q$. Theorem 5: Under the conditions of Theorem 4, if $M(\sigma)=\sum\mu(n)$, where the summation extends over all n's not exceeding N and falling into intervals of the form $cq<n\leqq(c+\sigma)q$, then $M(\sigma)=\sigma M(1)+O(N^{1+\epsilon}\Delta)$. The case $q=1$ seems to need some correction. *G. Pall* (Montreal, Que.).

Mardjanichvili, C. Sur la démonstration du théorème de Goldbach-Vinogradoff. C. R. (Doklady) Acad. Sci. URSS (N.S.) 30, 687–689 (1941). [MF 4471]

In this variant of Vinogradoff's proof of an asymptotic formula for the number of representations of odd numbers N as a sum of three primes, the approximation to the integrals of $S_\alpha^3e^{-2\pi i\alpha N}d\alpha$ on the major arcs $(\alpha=a/q+\theta/(q\tau)$, $q\leqq N/\tau)$ is effected by first decomposing

$$S_\alpha=\sum_{P<N}e^{2\pi i\alpha p}$$

into partial sums extended over primes in an arithmetical progression of common difference q. A formula due to A. Page [Proc. London Math. Soc. (2) 39, 116–141 (1935)] for the number of primes $qx+l$ is then applied, and the several terms so obtained are evaluated. For the minor arcs, Vinogradoff's work is used. The resulting error term is $O(N^2/(\log N)^{6-\epsilon})$, and is better than Vinogradoff's $O(N^2\log\log N/(\log N)^4)$. *G. Pall* (Montreal, Que.).

Wintner, Aurel. On a statistics of the Ramanujan sums. Amer. J. Math. 64, 106–114 (1942). [MF 5992]

Denote by $c_m(n)$ the Ramanujan sums. We have

$$c_m(n)=\sum_{\substack{0<r\leqq m\\(r,m)=1}}\cos(2\pi nr/m).$$

It is easy to see that the $c_m(n)$ are integers. For $n=1$ we have $c_m(n)=\mu(m)$. The author proves by using Tauberian theorems that the distribution function of $c_m(n)$ (with respect to m) exists for every n. For $n=1$ this is equivalent to the prime number theorem. *P. Erdös.*

Ingham, A. E. A Tauberian theorem for partitions. Ann. of Math. (2) **42**, 1075–1090 (1941). [MF 5817]

The author's principal aim is to deduce the asymptotic formulas

$$p(n)\sim e^{\pi(2n/3)^{\frac{1}{2}}}/4\sqrt{3}n,.\quad q(n)\sim e^{\pi(n/3)^{\frac{1}{2}}}/4\cdot 3^{\frac{1}{4}}n^{\frac{3}{4}},\quad n\to\infty,$$

from reasonably simple properties of the generating functions for $p(n)$ and $q(n)$, the number of unrestricted partitions of n and the number of partitions of n into unequal (or odd) parts, respectively. A Tauberian theorem for the integral $f(s)=\int_0^\infty e^{-su}dA(u)$ is first proved. This can be thought of as an interpolation between the theorems of Hardy-Littlewood type which conclude $A(u)\sim u^a$, and those of Wiener-Ikehara type which conclude $A(u)\sim e^{au}$ as $u\to\infty$. This general theorem is specialized to a theorem which is directly applicable to $p(n)$, $q(n)$ and other types of special partitions. In the application to $p(n)$, for example, the result is obtained from an elementary knowledge of the asymptotic behavior of the generating function $\sum_{n=0}^\infty p(n)z^n$, when $z=x+iy$ approaches the principal singularity $z=1$ in an arbitrarily large "Stoltz angle," $|y|\leqq\Delta(1-x)$, $0<\Delta<\infty$.

H. S. Zuckerman (Seattle, Wash.).

Lehner, Joseph. A partition function connected with the modulus five. Duke Math. J. **8**, 631–655 (1941). [MF 5948]

In this paper the author derives a convergent series for $p_a(n)$, $a=1$ or 2, the number of partitions of a positive integer into summands of the form $5l\pm a$. The method is that of Hardy and Littlewood in the improved form due to Rademacher [Amer. J. Math. **60**, 501–512 (1938)]. In Part I it is shown that the generating functions

$$F_a(x)=\prod_{m=0}^\infty(1-x^{5m+a})^{-1}\prod_{m=1}^\infty(1-x^{5m-a})^{-1}$$

transform according to the formula

$$F_a\left\{\exp\left(2\pi i\frac{h}{k}-2\pi\frac{z}{k}\right)\right\}$$
$$=\omega_a(h,k)\exp\left(\frac{\pi}{30k}\left(\frac{B}{z}-Az\right)\right)F_b\left\{\exp\left(2\pi i\frac{h'}{k}-\frac{2\pi}{kz}\right)\right\}$$

if $k\equiv0$ (mod 5) and in a more complicated way if $k\not\equiv0$ (mod 5). Here $\Re z>0$, h and k are coprime integers satisfying $0\leqq h<k$, h' is any fixed solution of $hh'\equiv-1$ (mod k); $b=1$ if $h\equiv\pm a$ (mod 5), $b=2$ if $h\equiv\pm2a$ (mod 5); and $B=6b^2-30b+25$. The functions $\omega_a(h,k)$ are certain roots of unity. Certain sums involving these roots of unity such as

$$\sum_h\omega_a(h,k)\exp(-2\pi ihnk^{-1})$$

have to be estimated more precisely than the trivial $O(k)$. This is done in Part II by reducing them to incomplete Kloosterman sums. Finally in Part III a convergent series and an asymptotic formula for $p_a(n)$ are derived following Rademacher's method. *R. D. James* (Madison, Wis.).

Hall, Marshall. A problem in partitions. Bull. Amer. Math. Soc. **47**, 804–807 (1941). [MF 5492]

Let $m\geqq n$ be positive integers, and let a set of m objects be "partitioned" into n non-void subsets $s_1, s_2, s_3, \cdots, s_n$. Such a partition determines a "partition matrix" (a_{ij}) such that $a_{ij}=1$ or 0 according as s_i and s_j have an object in common or not. Such a matrix is symmetric and its main diagonal consists entirely of 1's. The question proposed and answered in this paper is whether any given n-rowed matrix which is symmetric with 1's on the main diagonal and whose other elements are either 0 or 1 is, for some m, the partition matrix of a set of m objects into n subsets. The answer is in the affirmative if and only if $n<5$. *D. H. Lehmer.*

Tietze, Heinrich. Über Tripel konjugierter Partitionen. Abh. Math. Sem. Hansischen Univ. **14**, 273–284 (1941). [MF 5446]

This paper is concerned with the generalization to n dimensions of the notion of conjugate pairs of partitions of a number m arising from a two-dimensional graph. A set K of m lattice points in the x, y plane with positive integer coordinates represents a reduced (Ferrers) graph if for each point (x', y') of K the points (x, y) whose coordinates satisfy $0<x\leqq x'$, $0<y\leqq y'$ also belong to K. This graph, if read by rows, gives one partition of m and, if read by columns, gives another (possibly the same) partition. These two partitions have been called mutually conjugate. Clearly a given partition of m has one and only one partition conjugate to it.

In three dimensions one may, likewise, have a set K of m lattice points whose coordinates (x, y, z) are positive integers. These points are called reduced if for each point (x', y', z') of K the points (x, y, z) whose coordinates satisfy $0<x\leqq x'$, $0<y\leqq y'$, $0<z\leqq z'$ also belong to K. This reduced graph can be read by parallel planes in three ways to produce three conjugate partitions of m. This paper shows by means of examples that (contrary to the 2-dimensional case) in general the third conjugate of two arbitrarily chosen partitions of m is not uniquely determined, and, in fact, the number of available third conjugates is an unbounded function of m. The n-dimensional case ($n>3$) is similar.

D. H. Lehmer (Berkeley, Calif.).

Tietze, Heinrich. Systeme von Partitionen und Gitterpunktfiguren I. Rekursionsformeln. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1940**, 23–54 (1940). [MF 4574]

The problem under consideration may be described as follows. Let n and m be positive integers. Suppose we have given $n+1$ partitions $\mathfrak{A}^{(0)}, \mathfrak{A}^{(1)}, \cdots, \mathfrak{A}^{(n)}$ of the number m into m non-negative parts so that

$$\mathfrak{A}^{(\nu)}=(a_1^{(\nu)}, a_2^{(\nu)}, \cdots, a_m^{(\nu)}),\quad \nu=0, 1, 2, \cdots, n.$$

The problem is to determine the number

$$N=N(\mathfrak{A}^{(0)}|\mathfrak{A}^{(1)}|\cdots|\mathfrak{A}^{(n)})$$

of ways in which m^{n+1} numbers $g_{\mu_0\mu_1\cdots\mu_n}$ ($\mu_\nu=1, 2, \cdots, m$) may be assigned the values 0 or 1 in such a way that for each ν ($\nu=0, 1, \cdots, n$) and for each value of μ_ν the following equation holds:

$$\sum{}^{(\nu)}g_{\mu_0\mu_1\cdots\mu_n}=a_{\mu_\nu}^{(\nu)},$$

where $\sum^{(\nu)}$ is an n-fold multiple sum in which μ_ν is fixed. This problem, which is said to be of degree m and dimension $n+1$, is a generalization of the problem for $n=1$ which was considered in a previous paper [Monatsh. Math. Phys. **49**, 1–52 (1940); cf. these Rev. **1**, 323]. In the present paper is derived a complicated recursion formula for N which is illustrated by simple examples. The connection between the problem and the enumeration of certain lattice point figures in $n+1$ dimensional space is discussed in detail.

D. H. Lehmer (Berkeley, Calif.).

Watson, G. N. Proof of a conjecture stated by Mordell. J. London Math. Soc. 16, 157–166 (1941). [MF 6012]

Mordell remarked that a proof of the following conjecture is important in the discussion of lattice points connected with $f=x^p+y^p$: When $0<p<1$, the curve

$$\{(x+1)^p-1\}\{(y+1)^p-1\}=(1-x^p)(1-y^p)$$

possesses the property that any line parallel to $x+y=0$ cannot meet it at more than two points lying in the positive quadrant. The author demonstrates that the coefficients a_n $(n>0)$ of the power series

$$\{(1+\lambda+\xi)^p-1\}\{(1+\lambda-\xi)^p-1\} -\{1-(\lambda+\xi)^p\}\{1-(\lambda-\xi)^p\}=\sum_{n=0}^{\infty} a_n\xi^{2n}$$

are positive, when $\frac{1}{2}\leqq\lambda\leqq\frac{1}{2}(-1+\sqrt{5})$. The proof of the conjecture follows then quite simply. *C. L. Siegel.*

Mordell, L. J. Some results in the geometry of numbers for non-convex regions. J. London Math. Soc. 16, 149–151 (1941). [MF 6010]

Let L be a lattice $x=\alpha\xi+\beta\eta$, $y=\gamma\xi+\delta\eta$, where α, β, γ, δ are real numbers, $\alpha\delta-\beta\gamma=1$, and ξ, η run through all integer values. Let $f=f(x, y)$ be a homogeneous function of x, y, of positive dimension. The results are concerned with the best possible value of κ, independent of α, β, γ, δ, such that a point of L, other than $(0, 0)$, satisfies the inequality $|f(x, y)|\leqq\kappa$. Five applications are given: $f=x+y+\{\mu(x^2+y^2)+2xy\}^{\frac{1}{2}}$ $(0\leqq\mu<1)$; $f=\min(x+my, y+mx)$ $(3^{-\frac{1}{2}}\leqq m<1)$; $f=x^p+y^p$ ($p=\frac{1}{2}$ or $\frac{1}{3}$); $f=x^4-y^4$; $f=xy(x+y)$. The proofs are to be published in the Proc. London Math. Soc. *C. L. Siegel* (Princeton, N. J.).

Mordell, L. J. Lattice points in the region $|Ax^4+By^4|\leqq1$. J. London Math. Soc. 16, 152–156 (1941). [MF 6011]

In the case $f=x^4+y^4$, the best possible value of κ is $(6+4\sqrt{6})/15$ [cf. the preceding review]. By a geometric idea of Minkowski [Diophantische Approximationen, Teubner, Leipzig, 1907, pp. 47–58], the proof is reduced to the determination of the minimum value of $\alpha\delta-\beta\gamma$ under the conditions $\alpha^4+\gamma^4=1$, $\beta^4+\delta^4=1$, $(\alpha-\beta)^4+(\gamma-\delta)^4=1$, and the solution of this problem is then simplified by the theory of invariants of the binary quartic. *C. L. Siegel.*

Mordell, L. J. The product of homogeneous linear forms. J. London Math. Soc. 16, 4–12 (1941). [MF 5898]

$L_1, \cdots, L_n$ are n homogeneous linear forms in n variables with real coefficients and determinant 1. Let K_n be the lower bound of $|L_1L_2\cdots L_n|$, if the variables run over all lattice points except $0, \cdots, 0$. The following estimates are proved: $K_3\leqq4/27$, $K_4\leqq1/14.0\cdots$, $K_5\leqq132.4\cdots$. These are deduced from the theorem: Let there be a symmetric convex domain in the $(x_1, \cdots, x_{n-1})$ space with center at the origin and of volume $2^{n-1}V$, such that every point of it satisfies the inequality $|x_1\cdots x_{n-1}(x_1+\cdots+x_{n-1})|\leqq1$; then $K_n\leqq V^{-n/(n-2)}$. The proof of this theorem depends upon Minkowski's theorem concerning lattice points in convex bodies. The inequality $K_3\leqq4/27$ is less sharp than the best possible result $K_3\leqq1/7$ proved by Davenport [J. London Math. Soc. 13, 139–145 (1938)]. *C. L. Siegel.*

Žilinskas, G. On the product of four homogeneous linear forms. J. London Math. Soc. 16, 27–37 (1941). [MF 5901]

The method of the paper by Davenport mentioned in the above review is applied to the proof of the estimate $K_4\leqq3/20\sqrt{5}=1/14.9\cdots$, which is better than Mordell's result $K_4\leqq1/14.0\cdots$. *C. L. Siegel* (Princeton, N. J.).

Gelfond, A. On fractional parts of linear combinations of polynomials and exponential functions. Rec. Math. [Mat. Sbornik] N.S. 9 (51), 721–726 (1941). (Russian. English summary) [MF 5506]

Let

$$f(n)=\sum_{k=1}^{p} P_k(n)\alpha_k^n,$$

where the $P_k(z)$ are polynomials in z, and where the α_k are constants, all distinct, $|\alpha_k|>1$, $k=1, 2, \cdots, p$. The author supposes that the α_k and coefficients of the polynomials are (in general) complex numbers, and defines d_n to be the distance of $f(n)$ from the nearest integer of the Gaussian corpus. He proves (among other things) that there is a constant λ (depending only on the moduli of the α_k, the moduli of the coefficients of the P_k and the degrees of the polynomials) such that, if $d_n<\lambda n^{-\frac{1}{2}}$ for $n\geqq n_0$, then all α_k are algebraic integers; moreover, there is then a constant $\gamma>1$ for which $d_n<\gamma^{-n}$, $n\geqq n_1$. *D. C. Spencer.*

ANALYSIS

Theory of Probability

Copeland, A. H. Fundamental concepts of the theory of probability. Amer. Math. Monthly 48, 522–530 (1941). [MF 5715]

Copeland, Arthur H. Postulates for the theory of probability. Amer. J. Math. 63, 741–762 (1941). [MF 5622]

The collective theory of probability deals with 0, 1-sequences and thus, essentially, with subsets of the positive integers; in their quality of subclasses these constitute a Boolean algebra B in which union ($\vee$), intersection ($\cdot$) and complementation ($\sim$) have their set-theoretic meanings. Furthermore, if x and y are elements of B and correspond to the 0, 1-sequences $(x_1, x_2, \cdots)$ and $(y_1, y_2, \cdots)$, a new 0, 1-sequence $z:(z_1, z_2, \cdots)$ can be formed by taking in order each x_k for which $y_k=1$ (here $y\neq0$; and if only finitely many y_k's$=1$, cyclical repetition is resorted to). Notation: $z=x\subset y$. The collective theory can thus be stated in terms of $(B, \vee, \cdot, \sim, \subset)$.

This paper proceeds in the inverse order: It starts with a Boolean algebra B of undefined elements, and the logical operations $\vee$, $\cdot$, $\sim$ and $\subset$, where $x\subset y$ is read "x if y" and is described as a generalization of Russell's $y\subset x$ (y implies x). Then 15 postulates are introduced from which it is deduced (after 23 definitions and 43 theorems) that B, $\vee$, $\cdot$, $\sim$, $\subset$ are what we have described above and furnish the usual collective theories. In spite of their logical form the postulates do not seem destined to express laws of thought, but are posited solely in order to lead to the collective theory, with which they stand or fall epistemologically.

The author claims that his system differs from all others in that it furnishes (by $\subset$) the mathematical counterpart of an observation. But on the one hand, this claim is not in accordance with the facts of the literature [cf., e.g., B. O. Koopman, Bull. Amer. Math. Soc. 46, 763–774 (1940); cf.

these Rev. **2**, 106]; and on the other hand, in the present system $(x \subset y) \subset z$, $(x \subset z) \subset y$ and $x \subset (y \cdot z)$ are in general all distinct elements of B, and yet their conceptual counterparts (x if y) if z, (x if z) if y and x if x and y seem impossible to distinguish. *B. O. Koopman* (New York, N. Y.).

Ferrari, Esther. On Bertrand's paradox. II. Revista Union Mat. Argentina **7**, 74–80 (1941). (Spanish) [MF 5533]

The first part appeared in the same Revista **7**, 1–6 (1940); cf. these Rev. **2**, 227.

Mohr, Ernst. Bemerkungen zu Mises' Behandlung des Nadelproblems von Buffon. Deutsche Math. **6**, 108–113 (1941). [MF 5811]

The author objects to a remark by von Mises to the effect that the distribution associated with the Buffon needle problem depends on the manner of throwing the needle and can only be decided by experiment. The author asserts that such experimental (or statistical) determinations should be regarded as last resorts and applies the principle of insufficient reason to determine the distribution. He does not state whether this determination is independent of the manner of throwing the needle nor does he consider the possibility of obtaining other distributions by the same principle.
A. H. Copeland (Ann Arbor, Mich.).

***Fréchet, Maurice. Les probabilités associées à un système d'événements compatibles et dépendants. I. Événements en nombre fini fixe.** Actual. Sci. Ind., no. 859. Hermann et Cie., Paris, 1940. viii+80 pp.

This book is the first part of an exposition of the theory of dependent random events. The second part (stated to be ready for print) will be entitled "Cas particuliers et applications." The last part, to be written by another author, bears the title "Événements en nombre très grand ou infini."

Consider a class of random experiments with possible events $A_1, \cdots, A_m$. We are lead to consider various associated quantities such as: (i) the probability $p_{\alpha\beta\cdots\delta}$ of a simultaneous realization of $A_\alpha, A_\beta, \cdots, A_\delta$; (ii) the probability $p_{[\alpha\beta\cdots\delta]}$ of a realization of the events $A_\alpha, A_\beta, \cdots, A_\delta$ and only of these; (iii) the probability $P_{[r]}$ of the realization of exactly r among the m events A_α; (iv) the elementary symmetric functions of the $p_{\alpha\cdots\delta}$; (v) distribution functions, moments, etc., associated with these probabilities. The author first studies relationships between these quantities: how any of them can be expressed in terms of others, which among them can be arbitrarily prescribed, etc. An application is made to the problem of runs (iterations). The second chapter contains an exhaustive treatment of different inequalities between the above mentioned quantities, in particular Gumbel's generalizations of Boole's and Bonferroni's inequalities [Giorn. Ist. Ital. Attuari **9**, 1–38 (1938)].

Table of contents: I. Les relations d'égalité: loi de probabilité du nombre des événements concourants; probabilités d'événements dépendant moins simplement des A_j; calcul des probabilités de tous les événements qui sont fonctions d'un nombre fini d'événements A_j; formules symboliques; variables aléatoires; problème d'itération; probabilités conditionnelles; extension à plusieurs systèmes d'événements. II. Les relations d'inégalité: l'inégalité de Boole; généralisation de l'inégalité de Boole; conditions d'existence de systèmes d'événements correspondant à certaines probabilités données; systèmes d'événements ayant la même loi de répétition qu'un système donné.
W. Feller (Providence, R. I.).

Chung, Kai Lai. On the probability of the occurrence of at least m events among n arbitrary events. Ann. Math. Statistics **12**, 328–338 (1941). [MF 5418]

This paper is concerned with the probability that at least m of a set of n events will occur and with the corresponding probabilities associated with subsets of these events. The author obtains certain equalities and inequalities involving such probabilities. These results are generalizations of those obtained by Fréchet, Gumbel and others. The results are closely related to those previously announced by the author [C. R. Acad. Sci. Paris **210**, 620–621 (1940); cf. these Rev. **2**, 106]. *A. H. Copeland* (Ann Arbor, Mich.).

de Jongh, B. H. General minimum-probability theorem. Nederl. Akad. Wetensch., Proc. **44**, 738–743 (1941). (Dutch. German summary) [MF 5120]

Tchebycheff's theorem yields a positive minimum bound for the probability of a deviation of given magnitude, when for the set of variates the arithmetic mean and standard deviation are known and when the variates all satisfy some common distribution law or in more general terms when the distribution law for the set of variates is symmetric in these variates. By means of the interpretation of a distribution of masses (of unit total) along the arms of a balance, the author extends the theorem to arbitrary unsymmetric set-distributions. *A. A. Bennett* (Providence, R. I.).

Kawata, Tatsuo. The function of mean concentration of a chance variable. Duke Math. J. **8**, 666–677 (1941). [MF 5950]

If $\sigma(x)$ is the distribution function of a random variable X then

$$C(h)=(2h)^{-1}\int_{-\infty}^{+\infty}\{\sigma(x+h)-\sigma(x-h)\}^2dx$$

is called the "function of mean concentration of X." The author uses this function in order to derive some of P. Lévy's results concerning convergence and summability of series of chance variables. The advantage of using $C(h)$ rather than Lévy's "function of maximum concentration"

$$Q(h)=\max_{-\infty<x<\infty}\{\sigma(x+h+0)-\sigma(x-h-0)\}$$

is due to the possibility of applying Fourier analysis. It may be worth while noting that the author proves that $Q(2h)\geqq C(h)\geqq\frac{1}{2}Q^2(h)$. *M. Kac* (Ithaca, N. Y.).

Doeblin, W. Sur l'ensemble de puissances d'une loi de probabilité. Studia Math. **9**, 71–96 (1940). (French. Ukrainian summary) [MF 5257]

According to a terminology introduced by Khintchine two distribution functions (d.f.) $F(x)$ and $H(x)$ are said to belong to the same class c if there exist constants $a>0$ and b such that $F(x)=H(ax+b)$. The class c is called proper if it does not contain the unitary d.f. which vanishes for $x<0$ and equals 1 for $x\geqq 0$. The sequence of classes c_n converges to the proper class c if there exist $F_n(x)\epsilon c_n$ such that $F_n(x)\rightarrow F(x)\epsilon c$ at every point of continuity. The powers c^n of a class c^1 are defined as the classes containing the iterates $F_n=F_n*F_1$ of $F_1\epsilon c^1$. The present paper gives an exhaustive systematic study of convergence properties of the sequence

c^n. It contains many known theorems with improvements, sometimes of the proofs and sometimes of the results, and it adds several new results.

The study is based on the introduction of a metric in the space of all classes. This leads to various notions like class of accumulation, closed set, compact set of classes, derived set, etc. Let P be the set of the powers c^n, and P' its derived set. It is known [Khintchine, Rec. Math. [Mat. Sbornik] N.S. **1** (**43**), 71–120 (1937); P. Lévy, Théorie de l'addition des variables aléatoires, Paris, 1937] that P' is closed and contains only infinitely divisible distribution functions (i.d.d.f.). The author proves several related results; for example, he states conditions that the class of a given i.d.d.f. $G(x)$ be contained in P'. If $G(x)$ is the Gaussian d.f. this condition reduces to the condition that c^n converges to the class of $G(x)$ [Feller, Math. Z. **40**, 521–559 (1935); P. Lévy, l.c.]. If P' reduces to a single class this class is generated by a stable law. A striking new result is an example of a universal d.f., that is to say, of a class c^1 such that the set P' contains the classes of all i.d.d.f. The second part of the paper is devoted to questions of compactness for the sets P and P', and related problems; however, it is impossible to give an indication of all results here.

W. Feller (Providence, R. I.).

Gnedenko, B. V. Limit theorems for the maximal term of a variational series. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 7–9 (1941). [MF 5844]

Let $\xi_1, \xi_2, \cdots$ be a sequence of mutually independent random variables with the same distribution function $\mathfrak{F}(x)$, and let $\eta_n = \max\{\xi_1, \cdots, \xi_n\}$. It is known that, if there exist constants $a_n > 0$ and b_n such that the distribution function $F_n(x)$ of $(\eta_n - b_n)/a_n$ converges to $F(x)$, then there exist two constants $A > 0$ and B such that one of the following three relations holds: (i) $F(Ax+B) = e^{-e^{-x}}$; (ii) $F(Ax+B) = e^{-|x|^\alpha}$ for $x<0$ and $F(Ax+B)=1$ for $x>0$; (iii) $F(Ax+B)=0$ for $x<0$ and $F(Ax+B) = e^{-x^{-\alpha}}$ for $x>0$. Here α is a positive constant [Fréchet, Ann. Soc. Polon. Math. **6**, 93–116 (1927); R. A. Fisher and Tippett, Proc. Cambridge Philos. Soc. **24**, 180 (1928)]. The author states necessary and sufficient conditions for each of these cases. Furthermore, conditions are given that the sequence $\{\eta_n\}$ shall obey the law of large numbers. Proofs are not given.

W. Feller (Providence, R. I.).

Hartman, Philip and Wintner, Aurel. On the needle problem of Laplace and its generalizations. Bol. Mat. **14**, 260–263 (1941). [MF 5868]

Let there be given a lattice formed by the two sets of parallel lines $x=nh$, $y=mk$ ($m, n=0, \pm1, \pm2, \cdots, h>0, k>0$) in the (x, y) plane. The authors find the distribution function of the number of lattice points within a convex curve C thrown at random on the (x, y) plane.

M. Kac (Ithaca, N. Y.).

Daniels, H. E. The probability distribution of the extent of a random chain. Proc. Cambridge Philos. Soc. **37**, 244–251 (1941). [MF 4958]

By a random chain is meant what is usually called a random walk. Consider a chain having N links each of length a, the directions being independent random variables and all directions equally probable. The extent of the chain in any direction is defined as the shortest distance between a pair of planes perpendicular to that direction, such that the chain is contained entirely between them. The author considers chains in one or three dimensions, and derives the probability distribution and its mean value for the extent for an arbitrary N. The passage to the limit $N\to\infty$ is effected. It is shown that the extents in two perpendicular directions tend to be independently distributed. The problem is, of course, closely related with the differential (or difference) equation of heat diffusion.

W. Feller.

Cramér, Harald. Deux conférences sur la théorie des probabilités. Skand. Aktuarietidskr. **1941**, 34–69 (1941). [MF 5084]

The present conferences were held in Paris in 1937. In the first conference the author treats the familiar asymptotic expansions which he has established in Skand. Aktuarietidskr. **1928**, 13–74 and 141–180. A more recent representation of the subject is to be found in the author's "Random Variables and Probability Distributions" [Cambridge Tract no. 36 (1937)]. The second conference treats some special cases of the classical problem of ruin and the passage to the limit for a stochastic process depending on a continuous parameter; in particular, the application of the theory of stochastic processes to the so-called collective risk theory in insurance (originated by F. Lundberg and developed by the author) is stressed. In the meantime the subject has been treated exhaustively in a Stockholm dissertation by C. O. Segerdahl: "On Homogeneous Random Processes and Collective Risk Theory" [1939].

W. Feller (Providence, R. I.).

Fortet, R. Sur une suite egalement répartie. Studia Math. **9**, 54–70 (1940). (French. Ukrainian summary) [MF 5256]

This paper is concerned with the study of uniformly distributed sequences of the form

$$\text{(S)} \qquad x_n \equiv a^n x \pmod 1,$$

where $0 \leqq x_n < 1$; $n=0, 1, 2, \cdots$; $0\leqq x\leqq 1$, and $a>1$ is a fixed positive integer. Let $f(x) \in L$ $(0\leqq x\leqq 1)$ and put $\varphi_n(x) = (n+1)^{-1}\sum_{k=0}^{n} f(x_k)$. The author interprets Raikov's result that $\lim \varphi_n(x)$ exists almost everywhere (as $n\to\infty$) as a particular case of the strong law of large numbers for certain stationary chains [see A. Kolmogoroff, Rec. Math. [Mat. Sbornik] N.S. **2** (**44**), 367–368 (1937)]. Assuming that $f(x)$ satisfies Hölder's condition and employing the methods of Fréchet [Recherches théoriques modernes sur le calcul des probabilités, livre 2, p. 146. This is tome 1, fasc. 3 of Borel's Traité du calcul des probabilités, Paris, 1938] and Doeblin [Thèse, Univ. Paris, 1938 or Bull. Math. Soc. Roum. Sci. **39**, no. 1, 57–115; no. 2, 3–61 (1937); see, in particular, p. 98] the author proves the following two theorems. (I) The distribution function of

$$\left\{(n+1)\varphi_n(x) - (n+1)\int_0^1 f(x)dx\right\}(n+1)^{-\frac{1}{2}}$$

tends to a Gaussian distribution provided that the limit of

$$(n+1)^{-1}\int_0^1\left[\sum_{k=0}^{n} f(x_k) - (n+1)\int_0^1 f(x)dx\right]^2 dx$$

as $n\to\infty$ is different from 0. (II) Under the above condition the law of the iterated logarithm holds for the sum $\sum_{k=0}^{n} f(x_k)$. (I) is a generalization of an earlier result of the reviewer [Studia Math. **7**, 96–100 (1938); see also J. London Math. Soc. **13**, 131–134 (1938)]. The last part of the paper considers analogous questions for subsequences of (S).

M. Kac (Ithaca, N. Y.).

Kawada, Yukiyosi. On the measurable stochastic process. Proc. Phys.-Math. Soc. Japan (3) **23**, 512–527 (1941). [MF 5408]

The author formulates and discusses variants of the reviewer's definitions of a stochastic process depending on a continuous parameter [Trans. Amer. Math. Soc. **42**, 107–140 (1937)] and associated ideas. *J. L. Doob.*

Kitagawa, Tosio. The weakly contagious stochastic process which depends upon the Gaussian distribution. Mem. Fac. Sci. Kyûsyû Imp. Univ. A. **2**, 27–36 (1941). [MF 5636]

Previously [Mem. Fac. Sci. Kyûsyû Imp. Univ. A. **1**, 167–194 (1941); cf. these Rev. **2**, 230] the author has introduced the notion of a weakly contagious stochastic process. Here a special case is studied: in the notation of the previous review, it is assumed that all the g.d.f. $F_{n,k}(x)$ coincide with a common step function with jumps at $x=0$ and $x=1$ only. Let $S_n(m_n)=\sum_k^{m_n} Y_{n,k}$ and suppose that $m_n/n\rightarrow t$. It is shown that there exist numbers A_n and B_n (depending on t) such that $\{S_n(m_n)-A_n\}/B_n$ converges in probability. This leads to the definition of a weakly contagious stochastic process $\mathfrak{X}(t)$ of the generalized Gaussian type: for any sets of moments $s_1<t_1\leqq s_2<t_2\leqq \cdots \leqq s_k<t_k$ and for any set of k intervals (u_i, v_i) the probability that simultaneously $u_i<\mathfrak{X}(t_i)-\mathfrak{X}(s_i)<v_i$ is given by a k-dimensional Gaussian d.f., the parameters of which depend in a certain way on the contagion parameter and on the s_i, t_i; for details we must refer to the original. *W. Feller.*

Kitagawa, Tosio. The weakly contagious discrete stochastic process. Mem. Fac. Sci. Kyûsyû Imp. Univ. A. **2**, 37–65 (1941). [MF 5637]

In a previous paper [Mem. Fac. Sci. Kyûsyû Imp. Univ. A. **1**, 167–194 (1941); cf. these Rev. **2**, 230] the author has introduced various generalizations of the Pólya-Eggenberger scheme. Now he considers similar processes depending on a continuous time-parameter. The stochastic is purely discontinuous, the variable changing by jumps of the same amount (which means discrete events, as in the Poisson process). The process is contagious in the sense that the future probability relations depend on the whole prehistory of the process; this dependence is determined by a contagion function $\delta(t)$, which generalizes the contagion parameter of the review quoted above. The author also considers a locally contagious process where the probability relations for $t\geqq T$ depend only on the values of the stochastic variable for $T-t_0\leqq t\leqq T$, where t_0 is a constant. The limit theorems obtained before are generalized; furthermore, various probabilities are investigated, among which we mention the conditional probability $Q(t_1, t_2, t_3)$ that, a change having taken place during $(0, t_1)$, no change will occur during (t_1, t_2) but some change will occur during (t_2, t_3). The extremely complicated formulations make it impossible to give a full account of the author's results. *W. Feller.*

Theoretical Statistics

✶Darmois, G. Les mathématiques de la psychologie. Mémor. Sci. Math., fasc. 98, 51 pp. (1940). 25 fr.

This is an exposition of elementary mathematics and mathematical statistics which occur in connection with psychological testing. The material covered in the monograph is commonplace knowledge of a considerable body of American and British psychologists who have worked on problems arising in psychological testing. The first two chapters deal with statistical notions and definitions such as those for distribution, statistical independence, mean values, moments, correlation coefficient, etc. In the third chapter the author considers the implications of assuming a psychological characteristic to be expressible as a random variable consisting of the sum of two parts, one a true value (non-random) and the other a random error. He derives various well-known formulas such as the Spearman-Brown formula for the reliability of a test when extended in length, the Spearman formula for correcting correlation coefficients for attenuation, etc. In the next two chapters the author discusses the problem of describing and interpreting multiple measurements with particular emphasis on Hotelling's method of principal components and Spearman's one-factor hypothesis and their mathematical formulations. He mentions the multiple factor case and gives only the barest mathematical formulation of it. *S. S. Wilks.*

Dodd, Edward L. Some generalizations of the logarithmic mean and of similar means of two variates which become indeterminate when the two variates are equal. Ann. Math. Statistics **12**, 422–428 (1941). [MF 6052]

The logarithmic mean m of two positive numbers x and y is given by $(y-x)/(\log_e y-\log_e x)$. It arises, for instance, in the following problem: Suppose that $\phi(t)=(k/t)dt$ is the number of individuals with incomes lying between t and $t+dt$, where $x\leqq t\leqq y$. Then the total number of income receivers is given by $f=\int_x^y\phi(t)dt=k(\log y-\log x)$, and the total income is given by $g=\int_x^y t\phi(t)dt=k(y-x)$. Hence the logarithmic mean g/f of x and y is equal to the arithmetic mean of all the incomes. The author arrives at a generalization of the logarithmic mean in the following way: Let $\phi(t)$ be an integrable function defined over the interval $[a, a']$. Let furthermore $a<x_0<x_1<\cdots<x_n<a'$. Denote $\int_a^t\phi(t)dt$ by $\Phi(t)$ and $\int_a^t t\phi(t)dt$ by $\Psi(t)$. Then the mean M of $x_0, x_1, \cdots, x_n$ is defined by

$$M=\{\textstyle\sum_1^n k_r[\Psi(x_r)-\Psi(x_{r-1})]\}/\{\sum_1^n k_r[\Phi(x_r)-\Phi(x_{r-1})]\},$$

where the numbers k_r are positive. Some properties of M are investigated and also further generalizations are formulated. *A. Wald* (New York, N. Y.).

Haldane, J. B. S. The cumulants of the distribution of the square of a variate. Biometrika **32**, 199–200 (1941). [MF 5747]

Formulas are given which express the first few cumulants of the variable x^2 in terms of the cumulants of the variable x. *Z. W. Birnbaum* (Seattle, Wash.).

Bula, Clotilde A. Calculation of frequency surfaces. Experimental verification and comparison of the method of marginal functions with that of the 15 constants of Karl Pearson. Union Mat. Argentina, Publ. no. **10**, 109 pp. (1940) = Revista Union Mat. Argentina **6**, 109 pp. (1940). (Spanish. French summary) [MF 6086]

The author refers to a paper by C. E. Dieulefait [Biometrika **26**, 379–403 (1934)] in which a frequency function $f(x, y)$ is thus defined:

$$f(x, y)=\varphi(x)\rho(y)\sum_{j=0}\sum_{k=0}\omega_{j,k}X_j(x)Y_k(y),$$

in which $\varphi(x)$ and $\rho(y)$ are marginal totals as adjusted by the Pearson method; $X(x)$ and $Y(y)$ are polynomials or-

thogonal with respect to $\varphi(x)$ and $\rho(y)$; the $\omega_{j,k}$ are parameters to be determined. The author applies this method of marginal functions to the barometric heights at Southampton and Laudale studied by Pearson [Biometrika 17, 268–313 (1925), in particular p. 291]. The method of marginal functions is found to be unsuitable for these data. Considered in sections, the theoretical curves do not rise high enough at the modes; and the theoretical frequencies go down too far into negative values near and beyond the extremities of the actual given distribution, resulting in unreliable variances. This monograph contains 23 tables, 38 charts or diagrams, 74 numbered equations and 17 references. *E. L. Dodd* (Austin, Tex.).

Dieulefait, C. On a result of Prof. Beppo Levi and its relation to the problem of frequency surfaces. An. Soc. Ci. Argentina 129, 249–253 (1940). (Spanish) [MF 3777]

Remarks on the system

$$\int_{-\infty}^{\infty}\varphi(x, y)dy=p(x), \quad \int_{-\infty}^{\infty}\varphi(x, y)dx=q(y),$$

considered by Levi [Publ. Inst. Mat. Univ. Nac. Litoral 1, no. 1 (1939); these Rev. 2, 99]. The author is interested in the case where the unknown function $\varphi(x, y)$ is a probability density. By introducing the two-dimensional Gaussian distribution, he shows how to calculate the first few moments of a solution $\varphi(x, y)$. He suggests that this procedure should be useful in statistical problems. *R. P. Boas, Jr.* (Durham, N. C.).

Craig, Cecil C. A note on Sheppard's corrections. Ann. Math. Statistics 12, 339–345 (1941). [MF 5419]

Let m consecutive values of a discrete variate be grouped in each class interval of width k. Let $\bar{\nu}_{r0}$ denote the average value of rth central moments about the means of grouped distributions. With $s\neq 0$, let $\bar{\nu}_{rs}$ denote the average value of the product of the rth central moment of a grouped distribution by the sth power of the mean of the same grouped distribution. Then the corrected second moment is

$$\mu_2=\bar{\nu}_{20}+\bar{\nu}_{02}-k^2(1-1/m^2)/12.$$

Formulas are given for μ_3 and μ_4. As usual, $\mu_1=0$. Corrections are also found for seminvariants. *E. L. Dodd.*

Craig, C. C. On frequency distributions of the quotient and of the product of two statistical variables. Amer. Math. Monthly 49, 24–32 (1942). [MF 6142]

This is an expository treatment, based on elementary analysis, concerning the problem of determining the probability functions of the ratio and product of two random variables. The methods are applied to several special cases. In particular, the author gives the exact distributions of the ratio and of the product of two random variables jointly distributed according to the normal bivariate distribution. *S. S. Wilks* (Princeton, N. J.).

Krishnaswami Ayyangar, A. A. The triangular distribution. Math. Student 9, 85–87 (1941). [MF 6005]

The author gives for the triangular distribution a geometrical discussion of mode, median and arithmetic mean. Then he deduces a formula for the rth moment about the mean. In the symmetric case, this simplifies to $\mu_{2r+1}=0$, $\mu_{2r}=x_1^{2r}/(r+1)(2r+1)$, where x_1 is the half-range. For K_r the rth semi-invariant, he finds that, in the symmetric case, $K_{2r+1}=0$, $K_{2r}=(-1)^{r-1}B_r x_1^{2r}/r$, where B_r is the rth Bernoulli number. *E. L. Dodd* (Austin, Tex.).

Gordon, Robert D. Values of Mills' ratio of area to bounding ordinate and of the normal probability integral for large values of the argument. Ann. Math. Statistics 12, 364–366 (1941). [MF 5424]

The inequalities

$$\frac{x}{x^2+1}\cdot\frac{1}{(2\pi)^{\frac{1}{2}}}e^{-x^2/2}\leq\frac{1}{(2\pi)^{\frac{1}{2}}}\int_x^{\infty}e^{-t^2/2}dt\leq\frac{1}{x}\cdot\frac{1}{(2\pi)^{\frac{1}{2}}}e^{-x^2/2}$$

are proven. *Z. W. Birnbaum* (Seattle, Wash.).

Daniels, H. E. A property of the distribution of extremes. Biometrika 32, 194–195 (1941). [MF 5745]

It has been found empirically that the mean M and standard deviation σ of the distribution of the greatest value of X in a random sample of n from a normal distribution are approximately related as follows: $M=2\cot\frac{1}{2}\pi\sigma$, with a maximum error for n up to 1000 of not over 1.5 per cent. For n greater than 1000, the discrepancies are also found to be small. No theoretical basis is known by the author. *W. A. Shewhart* (New York, N. Y.).

Cochran, W. G. The distribution of the largest of a set of estimated variances as a fraction of their total. Ann. Eugenics 11, 47–52 (1941). [MF 4863]

The author first obtains the distribution of the largest of a set of correlated quantities [this distribution and more general distributions related to it were obtained by the reviewer, Ann. Math. Statistics 8, 159–176 (1937)]. This distribution is then used to obtain the distribution of the ratio of the largest of a set of independent variables that have the χ^2-distribution to the total of these variables. A table of the five per cent level of significance points for this ratio and a discussion of the limiting distribution of the ratio are presented. More general distributions associated with this problem can be obtained by using the paper of the reviewer referred to above. It may be noted that the power functions of these tests have also been obtained by the reviewer. *W. G. Madow* (Washington, D. C.).

Greville, T. N. E. The frequency distribution of a general matching problem. Ann. Math. Statistics 12, 350–354 (1941). [MF 5421]

If a card is drawn from each of two decks and the two cards thus obtained are of the same suit, this constitutes a matching. The decks may be unequal and the suits may be of unequal length. The author obtains a formula for the probability of r matchings in the course of exhausting one of the decks by such drawings. Greenwood obtained the first four moments of this distribution [Ann. Eugenics 10, 290–292 (1940); cf. these Rev. 2, 228], but Greville's interest is in simplifying the computation of the distribution. Greville applies his method to a problem of sampling connected with a contingency table. *A. H. Copeland.*

Cherian, K. C. A bi-variate correlated gamma-type distribution function. J. Indian Math. Soc. (N.S.) 5, 133–144 (1941). [MF 6133]

Let u_1, u_2, u_3 be independent statistical variables each obeying a χ^2-distribution law with n_1, n_2 and n_3 degrees of freedom, respectively. The author studies the joint distributions of $x=u_1+u_3$ and $y=u_2+u_3$, which have marginal

distributions of the χ^2-type. Moment characteristics and the moment generating function are obtained and special cases in which $n_1=2$ and in which $n_1=n_2=2$ are examined. The regression of means is linear in the second case and, if n_2 is even, in the first case. *C. C. Craig.*

Mood, A. M. On the joint distribution of the medians in samples from a multivariate population. Ann. Math. Statistics 12, 268–278 (1941). [MF 5412]

Suppose $(x_{1\alpha}, x_{2\alpha})$, $\alpha=1, 2, \cdots, 2n+1$, are elements of a sample drawn from a bivariate population having probability density function $f(x_1, x_2)$. Let $\bar{x}_1$ and $\bar{x}_2$ be the medians of the two sets of numbers $x_{1\alpha}$ and $x_{2\alpha}$ $(\alpha=1, 2, \cdots, 2n+1)$, respectively. The author gives an exact expression for the joint probability density function of $\bar{x}_1$ and $\bar{x}_2$ for any n. The limiting (as $n\rightarrow\infty$) probability density function of $y_1=(2n)^{\frac{1}{2}}a_1\bar{x}_1$ and $y_2=(2n)^{\frac{1}{2}}a_2\bar{x}_2$, where

$$a_1=\int_0^\infty f(0, x_2)dx_2+\int_{-\infty}^0 f(x_1, 0)dx$$

and

$$a_2=\int_0^\infty f(x_1, 0)dx_1+\int_{-\infty}^0 f(0, x_2)dx_2,$$

is shown to be a normal bivariate density function with a rather simple matrix of variances and covariances for certain weak regularity conditions on $f(x_1, x_2)$, including the condition that the population medians are zero. In the case where $f(x_1, x_2)$ is the normal bivariate distribution with correlation coefficient ρ, it is shown that the limiting value of the correlation coefficient between y_1 and y_2 (and hence between $\bar{x}_1$ and $\bar{x}_2$) is $(2/\pi)\sin^{-1}\rho$. The author extends his analysis to the case of k-dimensional distributions, showing that the limiting probability density function of k quantities similar to y_1 and y_2 is a normal multivariate density function. The elements of the matrix of variances and covariances for the case of three variables are given explicitly. *S. S. Wilks* (Princeton, N. J.).

Satterthwaite, Franklin E. Synthesis of variance. Psychometrika 6, 309–316 (1941). [MF 5632]

An estimate of the variance of the difference of the means of two independent samples is obtained by adding the estimates of the variance of each mean separately. In the case of samples from normal in which the two population variances σ_1^2 and σ_2^2 are unequal, this sum of estimates has a quite complicated distribution function. The author follows up a suggestion of B. L. Welch [Biometrika 29, 350–362 (1938); see, in particular, pp. 356, 357] that as a usable approximation we may assume that the sum of estimates is distributed as a single estimate with the degrees n of freedom determined from $(\sigma_1^2+\sigma_2^2)^2/n=\sigma_1^4/n_1+\sigma_2^4/n_2$, in which n_1 and n_2 are the distribution functions in the two samples, respectively. The main contribution here is a chart showing for a considerable range of values how well the approximate distribution proposed follows the exact distribution. There is some discussion of this point and of applications. *C. C. Craig* (Ann Arbor, Mich.).

Finney, D. J. The joint distribution of variance ratios based on a common error mean square. Ann. Eugenics 11, 136–140 (1941). [MF 5923]

Suppose v is a random variable distributed according to the χ^2-law with $2n$ degrees of freedom and $v_1, v_2, \cdots, v_k$ are independently distributed according to χ^2-laws with $2m$ degrees of freedom each, and are independent of v. Let $\varphi_i=v_i/v$, $i=1, \cdots, k$. The author shows that for any $a>0$ the value of

$$P(\varphi_i<a;\, i=1, 2, \cdots, k)=[M^k/(1+\lambda)^n]_{\lambda=0},$$

where M is the differential operator

$$e^{aD}(1-aD+a^2D^2/2!-\cdots+(-1)^{m-1}a^{m-1}D^{m-1}/(m-1)!)$$

and $D=\partial/\partial\lambda$. The meaning of k in M^k is to be taken in the usual operational sense. He also shows that the probability of exactly r of the φ's being less than a is

$$[\tbinom{k}{r}L^{k-r}M^r/(1+\lambda)^n]_{\lambda=0},$$

where $L=1-M$. Explicit formulas are given for $P(\varphi_i<na;\, i=1, \cdots, k)$ for $m=1$ and for $P(\varphi_i<na/2;\, i=1, \cdots, k)$ for $m=2$. For $m=1$; $2n=1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, \infty$; $k=1, 2, 3$, the value of a is determined so that the probability that exactly one of the k ratios exceeds a is 0.05. The author considers a method of approximating these values of a for $k=2$ and $k=3$ by replacing the actual distribution of the φ_i by what the distribution would be assuming independence. However, the validity of this method of approximation in this particular problem for general values of k remains an open question. *S. S. Wilks* (Princeton, N. J.).

Charnley, F. Some properties of a composite, bivariate distribution in which the means of the component normal distributions are linearly related. Canadian J. Research. Sect. A. 19, 139–151 (1941). [MF 5830]

Let X and Y be two random variables connected together by means of the equations $X=\xi+x$, $Y=a\xi+b+y$, where a and b are constant, ξ represents a random variable distributed over an interval P and x and y random variables (errors in determining ξ and $a\xi+b$), supposed to be independent of ξ but correlated among them, normal about expectations equal to zero, with (S.E.)'s σ_x and σ_y, respectively. The first part of the paper deals with the case where the distribution of ξ is uniform. It is shown that the straight line fitted by least squares to the regression of Y on X does not necessarily possess the slope a. In the second part of the paper the author drops the assumption of ξ being uniformly distributed and studies the relationships between the moments of all the variables involved. *J. Neyman* (Berkeley, Calif.).

Charnley, F. The variances of the means and the variance of the slope of the line of relation of a linear, composite, bivariate distribution. Canadian J. Research. Sect. A. 20, 6–9 (1942). [MF 5965]

Referring to an earlier paper of the author [see the preceding review], one finds that a linear, composite, bivariate distribution is one which is the sum of a number of bivariate distributions in which the means of the component distributions are collinear. It is further supposed that each component is a normal bivariate distribution with the variances and covariance the same for each component. If a sample is composed of a given number of drawings from each component, the obvious expression for the variance of the mean of the composite sample is found. If the sample contains equal numbers from each population component, an approximation, valid for large samples, is obtained for the variance of the slope of the line of means. *C. C. Craig* (Ann Arbor, Mich.).

Sarmanoff, O. W. Sur la corrélation isogène. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 28–30 (1941). [MF 5850]

It is said that between two random variables x and y there exists an isogenic correlation if the elementary relative probability law of any one of them, given the other, depends on a linear function of the first, so that $p(y|x)=af(ay+b)$ and $p(x|y)=\alpha\varphi(\alpha x+\beta)$, where a and b depend on x but not on y, while α and β depend on y but not on x. The author finds that, in the case of an isogenic correlation, the joint elementary probability law of x, y has the following general form

$$p(xy)=\text{const}\times\{Dx^2y^2+2Gx^2y+2Exy^2+Ax^2+Fy^2+2Hxy+2Bx+2Iy+1\}^c,$$

with the exponent $c<-1$ and the polynomial in brackets assuming only non-negative values. *J. Neyman.*

Samuelson, Paul A. A note on alternative regressions. Econometrica **10**, 80–83 (1942). [MF 5967]

This note refers to a previous paper of E. B. Woolley [Econometrica **9**, 38–62 (1941); these Rev. **2**, 235] in which a regression is fitted by minimizing the summed absolute values of the areas of the rectangles formed by the projections of each observation on the regression line. It is pointed out that it is nothing other than Frisch's "diagonal" regression and that whether this is a best regression line can only be decided in view of the assumptions made and the purpose in view. A list of properties which might be thought desirable in a regression line is set down and a unique characterization of the diagonal regression in terms of these is demonstrated. Finally it is brought out that previous methods of fitting regressions can be interpreted as minimizing areas. *C. C. Craig* (Ann Arbor, Mich.).

Baker, G. A. Linear regression when the standard deviations of arrays are not all equal. J. Amer. Statist. Assoc. **36**, 500–506 (1941). [MF 5881]

A method based upon an extension of a theorem of Markoff on least squares by F. N. David and J. Neyman [Statist. Research Mem. **2**, 105–116 (1938)] is presented for obtaining the "best" least square line of regression when the standard deviations of the arrays are not equal. An application made to a biometric problem shows that the proposed method of fitting may lead to an interpretation quite different from that obtained when allowance is not made for unequal variances of the arrays.
W. A. Shewhart (New York, N. Y.).

Bojarski, A. Sur la corrélation géométrique. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 159–164 (1941). (Russian. French summary) [MF 4512]

Let $f(x_1, x_2)$ be a function of two variables defined on the whole plane of x_1 and x_2. Fix a region R in the same plane and consider x_1 and x_2 as random variables uniformly distributed over R. Also fix two numbers X_1 and X_2. Then the correlation coefficient between the values of $f(x_1, x_2)$ and $f(x_1+X_1, x_2+X_2)$ is called the correlation coefficient between points (x_1, x_2) and (x_1+X_1, x_2+X_2) and denoted by $r(X_1, X_2)$. This correlation and various extensions, such as the average correlation between all pairs of points of a curve, etc., represent what is described as geometrical correlations.

In the case where the region R reduces to a linear interval, the correlation $r(x_1, x_2)$ depends only on the distance between the two points and is denoted by $r(\Delta)$. The author considers the correlations between three points A, B, C of which B lies between the two others. It is found that, if the partial correlation between A and C given B is equal to zero, then $r(\Delta)=\lambda^\Delta$. In the case of a two dimensional region R the problem of determining $r(\Delta)$ satisfying similar conditions is more complicated. The conditions considered by the author are: (1) the value of $r(x_1, x_2)$ depends only on $\Delta=(x_1^2+x_2^2)^{\frac{1}{2}}$; (2) if K and K' are two circumferences centered at a point A with radii a_1 and $a_2>a_1$, then the average partial correlation coefficient between A and points on K' given K is zero. It is found that, if correlations satisfying the above conditions exist at all, then the corresponding $r(\Delta)$ could not be at the same time non-negative and satisfy certain conditions of regularity. A number of results are given without proof. *J. Neyman.*

Kendall, M. G. The relationship between correlation formulae and elliptic functions. J. Roy. Statist. Soc. (N.S.) **104**, 281–283 (1941). [MF 6075]

It is well known that for the case of three variables there is a one-to-one correspondence between certain relations among partial and total correlation coefficients and certain relations between the sides and angles of spherical triangles. It is also well known that a duality exists between certain relations of angles and sides of spherical triangles and certain relations among Jacobian elliptic functions. The author has related the correlation coefficient formulas directly to formulas relating Jacobian elliptic functions by setting $r_{12}=\text{cn}\,u_3$, $r_{23}=\text{cn}\,u_1$, $r_{13}=\text{cn}\,u_2$. Formulas are then given for partial correlation coefficients, multiple correlation coefficients, etc., as functions of the elliptic functions.
S. S. Wilks (Princeton, N. J.).

Kendall, M. G. Proof of relations connected with the tetrachoric series and its generalization. Biometrika **32**, 196–198 (1941). [MF 5746]

By using the characteristic function associated with the normal bivariate distribution, the author verifies the known result that

$$\int_h^\infty\int_k^\infty f(x_1, x_2)dx_1dx_2=\sum_{r=0}^\infty \rho^r\tau_r(h)\tau(k),$$

where $f(x_1, x_2)$ is the normal bivariate probability function with unit variances, zero means and correlation coefficient ρ, while

$$\tau_r(x)=H_{r-1}(x)f(x)/(r!)^{\frac{1}{2}},$$

$H_r(x)$ being the rth Hermite polynomial and $f(x)$ being the normal distribution function with zero mean and unit variance. Convergence of the infinite series is established for $|\rho|\leqq 1$. The author discusses briefly the extension of this expansion to the case of several variables without going into any details. *S. S. Wilks* (Princeton, N. J.).

Geiringer, Hilda. A new explanation of nonnormal dispersion in the Lexis theory. Econometrica **10**, 53–60 (1942). [MF 5966]

In the classical Lexis theory supernormal and subnormal dispersions are explained by the heterogeneity of the populations. This theory presupposes independent populations. The author shows that in the case of dependent populations homogeneity will not produce normal dispersion. In fact positive correlation produces supernormal and negative correlation produces subnormal dispersion. Even in the heterogeneous case the correlation may be the dominant

factor. In certain cases the difference from normality may be larger than could possibly be explained by lack of homogeneity. *A. H. Copeland* (Ann Arbor, Mich.).

Hsu, Chung Tsi. Samples from two bivariate normal populations. Ann. Math. Statistics 12, 279–292 (1941). [MF 5413]

Given two samples of N and N' each from a normal bivariate population, the first with means ξ_1 and ξ_2, variances σ_1^2 and σ_2^2 and coefficient of correlation ρ, and with ξ_1', ξ_2', $\sigma_1'^2$, $\sigma_2'^2$, and ρ' the like parameters for the second, this paper is concerned with tests for the hypothesis that $\rho=\rho'$ under the conditions (1) that $\sigma_1=\sigma_2$ and $\rho_1'=\sigma_2'$, and (2) that $\sigma_1=\sigma_2$, $\sigma_1'=\sigma_2'$ and $\xi_1=\xi_2$, $\xi_1'=\xi_2'$. The likelihood ratio criteria for these tests are developed and for $N=N'$ the distribution functions of certain equivalent criteria (rather complicated to give here) are studied. For conditions (2) such a criterion is given and is shown to obey an F-distribution. The situation under conditions (1) is more complicated and after more extended analysis solutions are obtained expressed in general in quadratures or convergent hypergeometric series, though for special values of N they may reduce to elementary and finite forms. It is remarked that these results may readily be extended to the cases in which (3) $\sigma_1^2=k\sigma_2^2$ and $\sigma_1'^2=k'\sigma_2'^2$, and (4) $\sigma_1^2=k\sigma_2^2$, $\xi_1^2=k\xi_2^2$ and $\sigma_1'^2=k'\sigma_2'^2$, $\xi_1'^2=k'\xi_2'^2$, in which the k's are known. *C. C. Craig* (Ann Arbor, Mich.).

Hsu, P. L. On the limiting distribution of roots of a determinantal equation. J. London Math. Soc. 16, 183–194 (1941). [MF 6018]

Samples of $m_1, \cdots, m_k$ are taken respectively from k p-variate normal populations with means ξ_{ij} $(i=1, \cdots, p;$ $j=1, \cdots, k)$ and with a common set of covariances (and variances) σ_{ij} $(i, j=1, \cdots, p)$. The determination of the rank of the matrix $\|\xi_{ij}\|$ is important in linear discriminant analysis. If using $m_1, \cdots, m_k$ as weights we form column means from the elements of $\|\xi_{ij}\|$ and then the covariances of column elements ψ_{ij}, the problem is reduced to finding the number of positive roots of the determinantal equation $|\psi_{ij}-\lambda\sigma_{ij}|=0$, which is a special case of the wider problem of estimating the roots of this equation and of testing hypotheses concerning them. If $\bar{a}_{ij}$ and $\bar{b}_{ij}$ are the sample covariances corresponding to ψ_{ij} and σ_{ij}, respectively, in the present paper the limiting distribution of the non-zero roots of $|\bar{a}_{ij}-\rho\bar{b}_{ij}|=0$, as the sample sizes become infinite, is found, using results from his two earlier papers [Ann. Eugenics 9, 250–258 (1939); Biometrika 31, 221–237 (1940); see these Rev. 1, 248; 2, 111]. *C. C. Craig.*

Wald, Abraham. On the analysis of variance in case of multiple classifications with unequal class frequencies. Ann. Math. Statistics 12, 346–350 (1941). [MF 5420]

Consider N experiments liable to a two way classification according to values of two subscripts $i=1, 2, \cdots, p$ and $j=1, 2, \cdots, q$. Symbol (ij) will denote the category of experiments corresponding to a particular combination of the two subscripts. Let n_{ij} be the number of experiments of category (ij). Every particular experiment determines the values of three random variables x, ϵ and η, all assumed to be mutually independent and normally distributed with S.E.'s σ, σ' and σ''. It is known, however, that all the $n_{i.}=\sum_j n_{ij}$ experiments belonging to the categories corresponding to the same i determine the same value of the variable ϵ, which may be denoted by ϵ_i. Similarly, all the $n_{.j}=\sum_i n_{ij}$ experiments of the categories corresponding to the same value of the subscript j determine the same value of the variable η, which will be denoted by η_j. Although the values of the variables x, ϵ and η are determined by each experiment, these values are not directly observable. Instead we can observe only the values of sums $x+\epsilon+\eta=y$. The author gives the expressions of two functions of the observable y's representing the confidence limits for the ratio σ'/σ. *J. Neyman* (Berkeley, Calif.).

Daly, Joseph F. A problem in estimation. Ann. Math. Statistics 12, 459–461 (1941). [MF 6055]

Let an individual E be characterized by the variates $y^1, \cdots, y^p, y^{p+1}, \cdots, y^{p+t}$ $(p>t)$, of which the first p are observable and the last t are not. However, the linear regressions of the p observable y's on the t unobservable ones are given, together with the information that the $p+t$ y's are normally distributed and that further each y^i $(i=1, \cdots, p)$ for given y^r $(r=p+1, \cdots, p+t)$ is independently and normally distributed about its regression with a common variance σ^2. It is required to estimate the means, variances and covariances of y^r, with no parameters in the distribution laws known except the coefficients which specify the given regressions. The means sought are easily estimated by maximum likelihood. For the variances and covariances, the y^r are integrated out of the joint distribution of all $p+t$ y's and the result is identified with given form of distribution of y^i. Then again applying the method of maximum likelihood to this result the required estimates are obtained. *C. C. Craig* (Ann Arbor, Mich.).

Lawley, D. N. Further investigations in factor estimation. Proc. Roy. Soc. Edinburgh. Sect. A. 61, 176–185 (1942). [MF 6228]

The author returns again to the use of maximum likelihood methods of factor estimation in factor analysis previously considered by him [Proc. Roy. Soc. Edinburgh 60, 64–82 (1940); see these Rev. 2, 110]. He shows that by these methods it is possible to estimate simultaneously the factor loadings of the tests, the factor measures of the individuals taking the tests and the error variances, under the assumption that these last quantities depend on only the corresponding tests. An iterative process given by numerical estimates may be obtained and this is illustrated and the results compared with those given by other methods. *C. C. Craig* (Ann Arbor, Mich.).

Young, L. C. On randomness in ordered sequences. Ann. Math. Statistics 12, 293–300 (1941). [MF 5414]

Given a finite ordered sequence $X_1, X_2, \cdots, X_n$ of n measurements, the author chooses the statistic

$$C=1-\sum_1^{n-1}(X_i-X_{i+1})^2\left\{2\sum_1^n(X_i-\bar{X})^2\right\}^{-1}$$

as the basis of a test of randomness, where $\bar{X}$ is the average. Significance levels of the absolute value of C are given for samples from 8 to 25. The field of application of the C test is briefly discussed and an illustration from the field of quality control is given. *W. A. Shewhart.*

Paulson, Edward. On certain likelihood-ratio tests associated with the exponential distribution. Ann. Math. Statistics 12, 301–306 (1941). [MF 5415]

In this paper the exponential elementary probability law $\sigma^{-1}e^{-(x-B)/\sigma}$, $B\leqq x<\infty$, is considered and the power functions of the likelihood-ratio tests are calculated for the

following three hypotheses: (1) the hypothesis that the location parameter B is equal to some specified value, it being assumed that the scale parameter σ is known; (2) the hypothesis that B is equal to some specified value when nothing is known about σ; (3) the hypothesis that two samples are drawn from exponential distributions with the same location parameter, assuming it is known that the samples must have come from two exponential distributions with the same scale parameter. It is furthermore shown that all three tests are completely unbiased. *A. Wald.*

Neyman, J. Fiducial argument and the theory of confidence intervals. Biometrika 32, 128–150 (1941). [MF 5741]

The author presents his reasons for believing that the theory of fiducial probability is distinct from that of confidence intervals. He shows that the two theories differ in their basic concepts and that fiducial limits may in some instances, but will not in others, be numerically the same as confidence limits. *W. A. Shewhart* (New York, N. Y.).

Fisher, R. A. The asymptotic approach to Behrens's integral, with further tables for the d test of significance. Ann. Eugenics 11, 141–172 (1941). [MF 5924]

Behrens' integral is used for obtaining fiducial limits for the difference of the means of two normal populations when nothing is known about the variances of these populations. In this paper the author gives an asymptotic approximation to Behrens' integral and constructs some tables. The validity of the use of Behrens' integral for the problem under consideration is based on the author's theory of fiducial inference, which has been a controversial issue among statisticians in recent years. [See in this connection the paper reviewed above.] *A. Wald* (New York, N. Y.).

Aroian, Leo A. A study of R. A. Fisher's z distribution and the related F distribution. Ann. Math. Statistics 12, 429–448 (1941). [MF 6053]

The distribution of Fisher's z, particularly its moment and semi-invariant characteristics, are studied in detail together with approximations valid for larger values of the parameters. The nature of the approach of this distribution function to the normal frequency law is investigated and there is considerable material on the usefulness, for certain ranges, of using the Pearson type III function, the logarithmic-normal function or the Gram-Charlier Type A series for the approximate calculation of the areas critical for significance tests under the z-distribution curve. There is a very brief section devoted to the distribution of Snedecor's F. *C. C. Craig* (Ann Arbor, Mich.).

Jeffreys, Harold. Some applications of the method of minimum χ'^2. Ann. Eugenics 11, 108–114 (1941). [MF 5922]

Sawkins, D. T. Remarks on goodness of fit of hypotheses and on Pearson's χ^2 test. J. Proc. Roy. Soc. New South Wales 75, 85–95 (1941). [MF 5453]

General discussion of criteria for testing statistical hypotheses. A statistical hypothesis H is to be rejected if the observations reveal some characteristics which, from the point of view of the hypothesis H, are "rare." For example, one may try the χ^2, but, if it happens to have a value which is not "rare," one should consider the distribution of signs of the variables observed; if these are found to conform with H, one should consider the values of the smallest and of the largest observations, etc. In other words, the suggestion is that the criterion for testing H should be selected out of an unlimited class after the observations are obtained and that the credence should be given to the one which is the least favorable to the hypothesis tested. It was shown, however [J. Neyman, Lectures and Conferences on Mathematical Statistics, Washington, D. C., 1938, pp. 33–44], that, if this principle is adopted, then no particular calculations to test statistical hypotheses are necessary because, with a certain amount of skill, every hypothesis will have to be rejected on any set of data. The paper contains also a deduction of the χ^2 distribution. *J. Neyman.*

Thompson, Catherine M. Table of percentage points of the χ^2 distribution. Biometrika 32, 187–191 (1941). [MF 5744]

Let $f_\nu(\chi^2)d\chi^2$ be the differential element for the χ^2 distribution with ν degrees of freedom. R. A. Fisher has calculated χ_0^2 such that $\int_{\chi_0^2}^{\infty} f_\nu(\chi^2)d\chi^2 = P$ for $P=0.99$, 0.98, 0.95, 0.90, 0.80, 0.70, 0.50, 0.30, 0.20, 0.10, 0.05, 0.02, 0.01 and for $\nu=1(1)30$. The present author extends these tables to include $P=0.995$, 0.99, 0.975, 0.95, 0.90, 0.75, 0.50, 0.25, 0.10, 0.05, 0.025, 0.01, 0.005 and $\nu=1(1)30(10)100$. *S. S. Wilks* (Princeton, N. J.).

Merrington, Maxine. Numerical approximations to the percentage points of the χ^2 distribution. Biometrika 32, 200–202 (1941). [MF 5748]

A table of percentage points of χ^2 for numbers of degrees of freedom not less than 30 is computed from an approximate formula given by R. A. Fisher, and from another one due to E. B. Wilson and M. M. Hilferty. A comparison with the exact values shows that the second formula yields a better approximation. *Z. W. Birnbaum.*

Kosambi, D. D. A bivariate extension of Fisher's z test. Current Sci. 10, 191–193 (1941). [MF 5828]

S. S. Wilks [Biometrika 24, 471–494 (1932)] has studied the ratio of two independently distributed generalized sample variances and derived its exact sampling distribution for bivariate populations. In this note the author points out that a simple elementary function of this ratio has Fisher's z-distribution and therefore statistical tests based on this ratio can easily be carried out by using Fisher's tables. *A. Wald* (New York, N. Y.).

Kosambi, D. D. Correlation and time series. Current Sci. 10, 372–374 (1941). [MF 5827]

Let s be a variate the value of which is observed at different time points. The author points out that the correlation coefficient r between s and the time variable t is not a proper measure for the dependence between these variables, since s and t do not form a bivariate normal population. Instead of r, the author proposes the use of some other measure, to which, in the opinion of the reviewer, similar criticism can be applied. *A. Wald* (New York, N. Y.).

Ono, Katudi. Über eine Art der Ausgleichung der statistischen Reihen. Jap. J. Math. 17, 513–515 (1941). [MF 5643]

***Davis, H. T. The statistics of time series.** Northwestern University Studies in Mathematics and the Physical Sciences, no. 1: Mathematical Monographs, vol. 1, pp. 45–85. Graduate School, Northwestern University, Evanston, Ill., 1941. $2.25.

This publication contains a brief discussion of several important topics of time series analysis, such as harmonic

analysis and tests of significance, serial and auto-correlations, random series, etc. Although the treatment of these problems is far from complete, the reader will find there many interesting statements, especially concerning inverse serial correlations and certain properties of random series.

A. Wald (New York, N. Y.).

✶Davis, Harold T. The Analysis of Economic Time Series. The Cowles Commission for Research in Economics, Monograph No. 6. Principia Press, Inc., Bloomington, Ind., 1941. xiv+620 pp. $5.00.

The first part of the book (roughly 6 chapters) contains a general outline of various theories of time series and the techniques usually employed for their analysis. Special economic time series are extensively used as illustrative material; on the whole, however, the exposition is of a general statistical character. The topics discussed include the following: History of the problem [chap. 1]. Harmonic analysis, in particular the construction of periodograms; approximation by orthogonal functions [chap. 2]. Serial and lag correlations, inverse correlation, etc. Particular attention is paid to Yule's theory of random variation in connection with the erratic-shock theory of economic time series. Also Slutsky's so-called sinusoidal limit law for accumulated random series and its economic implications are treated in some detail [chap. 3 and 4]. Next we find various tests of significance in harmonic analysis and an exposition of factor analysis [chap. 5]. Finally the author treats various types of trends, in particular the logistic trend.

The second part of the book is devoted mainly to economic theory. The preceding tools are applied to an exceedingly large amount of economic material and the significance of the results is discussed. The different theories of cycles form, of course, the first main topic. Here, in addition to the common theories, the author discusses also the effect of a lag-hysteresis of the type occurring in the Volterra-Lotka theory of struggle for existence. Other topics include the nature of wealth, the dynamics of trends, the possibility and significance of forecasting.

One of the most pleasant features of the book (besides its wide range) is the clear recognition that the employment of a specific mathematical tool in itself implies an economic assumption. Thus the final conclusions will depend on the tools employed and on the postulates which underlie these tools. This recognition necessitates a careful mathematical analysis. It is needless to point out, however, that the book is not written for mathematicians; hence no complete mathematical theory could be given and the book has necessarily become, in part at least, descriptive in character. Occasionally the conditions of validity of a formula or its approximative character are not made quite clear. This, however, will not affect the value of the book for economists and for statisticians in general. *W. Feller.*

✶Wallis, W. Allen and Moore, Geoffrey H. A Significance Test for Time Series. National Bureau of Economic Research, Technical Paper no. 1. New York, 1941. xii+59 pp. $.50.

In the summary, the authors state: "A simple and economical test of significance for time series (and other data in which the order of appearance is essential), which makes no assumption about the fundamental probability distribution, may be based on the frequency distribution of sequences of like sign in the first differences. In a series of N independent random observations the expected number of completed runs of d in the signs of the first differences is

$$\frac{2(d^2+3d+1)(N-d-2)}{(d+3)!}.$$

. . . These expectations [for various d] may be compared with the observed frequencies by the usual method of summing the ratios of the squared deviations to the expectations. The sum $\cdots$ is denoted by $\chi_p{}^2\cdots$. The tail of the distribution of $\chi_p{}^2\cdots$ is well described by the χ^2 distribution for $2\frac{1}{2}$ degrees of freedom $\cdots$. Although these empirical distributions seem adequate for practical work, a rigorous derivation of the true sampling distribution is much to be desired." In Table 5, the above distribution is given for $\chi^2 \geqq 5.448$, which corresponds to $P=0.10$. A summary appears in the J. Amer. Statist. Assoc. **36**, 401–409 (1941).

E. L. Dodd (Austin, Tex.).

Applications of the Theory of Probability

Bernstein, S. Solution of a mathematical problem connected with the theory of heredity. Ann. Math. Statistics **13**, 53–61 (1942). [MF 6383]

Translation of the author's paper originally published in the Ann. Sci. Ukraine **1**, 83–114 (1924). A resume appeared in the C. R. Acad. Sci. Paris **177**, 528–531, 581–584 (1923).

Scholz, Edmund. Ein methodischer Beitrag zur Berechnung des Erbgefüges. Deutsche Math. **6**, 100–104 (1941). [MF 5809]

Matrices are sometimes used for the practical computation of the hereditary structure of successive generations. According to the author it is simpler and more natural to use quadratic and linear forms. He gives an elementary exposition of the theory in these terms. *W. Feller.*

Scholz, Edmund. Die Auflösung eines gewissen linear homogenen Systems von Rückschlussformeln und ihre Anwendung auf Probleme der Inzucht. Deutsche Math. **6**, 104–107 (1941). [MF 5810]

The author is concerned with the recursive system $x_i^{(n+1)}=\sum a_{ik}x_k{}^{(n)}$ $(i, k=1, \cdots, r)$, encountered in the theory of heredity. Using the characteristic roots and vectors of the matrix (a_{ik}) he obtains a direct representation of $x_i{}^{(n)}$ in terms of the $x_k{}^{(0)}$. [This representation could be simplified, and leads then to a well-known expression which follows directly from the canonical representation of (a_{ik}).]

W. Feller (Providence, R. I.).

✶Jecklin, Heinrich. The theory of probability in insurance. Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 71–115. Orell Füssli, Zurich, 1941. (Paper in German, French, Italian, English)

Subject I of the congress mentioned in the title is "The theory of probability in insurance." In the present paper "an attempt is made to give a summary of the papers dealing with this subject [cf. the following reviews] and of the results arrived at, arranging these according to some systematic classification."

✶**Baptist, Jean. Le calcul des probabilités dans le domaine de l'assurance.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 117–136. Orell Füssli, Zurich, 1941. (German, Italian, English summaries)

✶**Shannon, Simon. Some assumptions and hypotheses underlying actuarial calculations.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 137–158. Orell Füssli, Zurich, 1941. (German, French, Italian summaries)

✶**Boschan, Paul. Some considerations concerning probability in actuarial science and the foundation of the extended life table.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 159–169. Orell Füssli, Zurich, 1941. (German, French, Italian summaries)

✶**Hammon, Philip H. and Clarke, R. D. Some effects upon insurance problems of modern criticisms of the frequency theory of probability.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 207–224. Orell Füssli, Zurich, 1941. (German, French, Italian summaries)

✶**Redington, Frank M. and Michaelson, Ronald L. An aspect of the "a priori" probability theory of mortality.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 225–239. Orell Füssli, Zurich, 1941. (German, French, Italian summaries)

✶**Vajda, Stefan. Die erweiterte Sterbetafel und ihre wahrscheinlichkeitstheoretische Verwendung.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 241–251. Orell Füssli, Zurich, 1941. (French, Italian, English summaries)

✶**de Finetti, Bruno. Il calcolo delle probabilità nel dominio dell'assicurazione.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 253–261. Orell Füssli, Zurich, 1941. (German, French, English summaries)

✶**ten Pas, W. G. J. Wahrscheinlichkeitsrechnung und Statistik in der Versicherungsmathematik.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 263–283. Orell Füssli, Zurich, 1941. (French, Italian, English summaries)

✶**Hagstroem, K.-G. Quelques réflexions sur le rôle de la théorie des probabilités dans l'assurance pratique.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 321–328. Orell Füssli, Zurich, 1941. (German, Italian, English summaries)

✶**Jecklin, Heinrich. Ist die Anwendbarkeit der Wahrscheinlichkeitstheorie in der Lebensversicherung besser fundiert als in der Sachversicherung?** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 377–393. Orell Füssli, Zurich, 1941. (French, Italian, English summaries)

✶**Lah, Ivo. Wahrscheinlichkeitsrechnung und Versicherungswesen.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 409–434. Orell Füssli, Zurich, 1941. (French, Italian, English summaries)

✶**Berger, Alfred. Welche Hypothesen liegen der Versicherungsmathematik zugrunde und wie kann die Anwendung der Wahrscheinlichkeitstheorie und der Risikotheorie im Versicherungswesen begründet werden?** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. IV, pp. 9–26. Orell Füssli, Zurich, 1941. (French, Italian, English summaries)

✶**Kolodziejczyk, Stanislaw. Sur les événements à probabilité élevée.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 309–319. Orell Füssli, Zurich, 1941. (German, Italian, English summaries)

Let p denote the probability that an insured object of a certain kind will be destroyed by fire during one year, n the number of such objects under risk, and ν the number of those actually destroyed in one year. Then $|\nu/n-p|/p=\epsilon$ is the relative error of the frequency ν/n and, for large n, the approximate equality holds

$$P\{\epsilon\leqq\lambda((1-p)/np)^{\frac{1}{2}}\}\approx(2\pi)^{-\frac{1}{2}}\int_{-\lambda}^{+\lambda}e^{-x^2/2}dx=f(\lambda).$$

For any given "coefficient of confidence" α it is possible to determine λ so that $f(\lambda)=1-\alpha$, and then to assume conventionally that for practical purposes ϵ never exceeds $\lambda((1-p)/np)^{\frac{1}{2}}$. Since, for a fixed α, and hence fixed λ, the quantity $\lambda((1-p)/np)^{\frac{1}{2}}$ decreases as p increases, it is stated that the relative error ϵ tends to be smaller with increasing p. By similar arguments it is shown that the relative mean square risk and certain safety coefficients for premiums decrease as p increases. Analogous statements are proven for more complicated groups of risks.

Z. W. Birnbaum (Seattle, Wash.).

✶**Lukacs, Eugen. Wahrscheinlichkeitstheoretischer Aufbau der Theorie des mittleren Risikos.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 171–205. Orell Füssli, Zurich, 1941. (French, Italian, English summaries)

In the actuarial theory it is usually assumed that for every x there is a probability q_x that an individual of age x will die within the next year. Death or survival of an individual is therefore considered as a simple alternative with known probabilities. The author makes the more general assumption that for each age x there is a probability distribution $v_x(s, a)$ which indicates the probability that a individuals will die within the next year out of s individuals living at age x, for every x, s and $0\leqq a\leqq s$. For those probability distributions only one restrictive assumption is made, namely, that for every x the ratio of the mathematical expectation of a to the number s of those alive is independent of s. Under those very general assumptions the author develops not only the theory of premiums and reserves but also a theory of the mean square risk including theorems analogous to those of Hattendorf and Hausdorff in the classical theory. The formulas of this generalized risk theory contain explicitly the variances of the probability distributions $v_x(s, a)$, and this suggests the possibility of

determining those variances empirically and thereby adapting the risk theory to the experience better than it has been possible in the theory based on the probabilities q_x.
Z. W. Birnbaum (Seattle, Wash.).

***Jacob, M. Su di un metodo d'approssimazione per il calcolo del rischio quadratico medio.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 285–307. Orell Füssli, Zurich, 1941. (German, French, English summaries)

Let $\gamma(\nu)$ be a positive function and $f(\nu)$ and $\rho(\nu)$ two positive and nonincreasing functions. Putting

$$F=\left(\sum_{\nu=1}^{n} f(\nu)\gamma(\nu) \sum_{\nu=1}^{n} \rho(\nu)\gamma(\nu)\right)\Big/\sum_{\nu=1}^{n} \gamma(\nu),$$

one has the inequality $\sum_{\nu=1}^{n}\rho(\nu)f(\nu)\gamma(\nu)\geqq F$. In case $\rho(\nu)$ is a positive nonincreasing function whereas $f(\nu)$ is a positive nondecreasing function one has the inequality

$$\sum_{\nu=1}^{n}\rho(\nu)f(\nu)\gamma(\nu)\leqq F.$$

Using these inequalities, which are due to J. F. Steffensen [see, for example, Some Recent Researches in the Theory of Statistics and Actuarial Science, Cambridge University Press, Cambridge, England, 1930, p. 29], the author derives upper and lower limits for the mean risk as given by Hattendorf's formula

$$M^2(0, n)=\sum_{\nu=0}^{n-1} (l_{x+\nu}/l_x)q_{x+\nu}p_{x+\nu}v^{2(\nu+1)}\{1-{}_{\nu+1}V\}^2.$$

Putting first $\gamma(\nu)=v^{2(\nu+1)}$, $\rho(\nu)=(1-{}_{\nu+1}V)^2$, $f(\nu)=p_{x+\nu}d_{x+\nu}/l_x$, an upper limit is obtained; substituting then $\gamma(\nu)=1$, $\rho(\nu)=(1-{}_{\nu+1}V)^2$, $f(\nu)=(p_{x+\nu}d_{x+\nu}/l_x)v^{2(\nu+1)}$, a lower limit for $M^2(0, n)$ is found. The assumptions of Steffensen's inequalities are fulfilled up to an age of about 70. Using these limits the author derives an approximation to the mean risk of an endowment insurance in case of sinking premiums if the mean risk for the case of equal premiums is known.
E. Lukacs (Baltimore, Md.).

***Andersen, N. E. De la durée moyenne d'une série de prestations en capitaux arbitraires en fonction du taux instantané d'intérêt et des demi-invariants de la répartition.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. III, pp. 437–457. Orell Füssli, Zurich, 1941. (German, Italian, English summaries)

Let $S_1, S_2, \cdots, S_k$ be a number of arbitrary payments due at the end of certain equal intervals. Let δ be the force of interest, $v=e^{-\delta}$, and $S=\sum_{\nu=1}^{k}S_\nu$ the sum of the k payments. The mean duration $n(\delta)$ is defined by $\sum_{\nu=1}^{k}S_\nu e^{-\delta\nu}=Se^{-\delta n(\delta)}$. Using the inequalities of Steffensen and Jensen [Steffensen, J. Inst. Actuar. 51, 274–297 (1919); Skand. Aktuarietidskr. 8, 137–147 (1925)] the author discusses the function $n(\delta)$ and shows that $n(\delta)$ is a decreasing function of δ whatever the sums S_ν be. He also shows that

$$n(\delta)=\sum_{i=1}^{\infty} (-1)^{i-1}(\mu_i/i!)\delta^{i-1},$$

where the μ_i are the semi-invariants of the distribution of the sums S_ν.
E. Lukacs (Baltimore, Md.).

***Haferl, Eduard. Die Bestimmung der Selbstbehalte in der Lebensversicherung.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 349–376. Orell Füssli, Zurich, 1941. (French, Italian, English summaries)

***Nolfi, Padrot. Die jährlichen Sterblichkeitsschwankungen und ihre wahrscheinlichkeitstheoretische Erfassung.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 395–408. Orell Füssli, Zurich, 1941. (French, Italian, English summaries)

***Riebesell, Paul. Die mathematischen Grundlagen der Sachversicherung.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. IV, pp. 27–36. Orell Füssli, Zurich, 1941. (French, Italian, English summaries)

Report on some recent papers, in particular, one by W.-G. Ackermann [Schr. Math. Inst. u. Inst. Angew. Math. Univ. Berlin 4, 211–255 (1939); cf. these Rev. 1, 251].
W. Feller (Providence, R. I.).

***Lundberg, Ove. On the importance of regrading risk premiums in voluntary sickness and accident insurance. A theoretical basis for regrading.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. II, pp. 543–560. Orell Füssli, Zurich, 1941. (German, French, Italian summaries)

The author gives a summary of the theory of a special type of Markoff process developed in his book "On Random Processes and their Applications to Sickness and Accident Insurance" [Uppsala, 1940; cf. these Rev. 2, 230]. Some technical questions are also treated.
W. Feller.

Stern, E. Leibrenten und veränderliche Todesfallversicherungen. Verzekerings-Arch. 22, 285–318 (1941). [MF 6084]

***Zwinggi, Ernst. Study of internal variation in groups of persons.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. III, pp. 263–303. Orell Füssli, Zurich, 1941. (Paper in German, French, Italian, English)

A general report.

***Hadwiger, Hugo und Wegmüller, Walter. Entwicklung und Umschichtung von Personengesamtheiten.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. III, pp. 369–386. Orell Füssli, Zurich, 1941. (French, Italian, English summaries)

Consider a self-renewing aggregate in which all members are subjected to the same mortality force $\mu(x)$, where x is the member's age in the population. Let the number of members of age x at time t be given by $S(x, t)$. The authors show that $S_x+S_t+\mu S=0$, whence, after some computations, they find the obvious and well-known expression

$$S=f(t-x)\exp\left\{-\int_0^x \mu(s)ds\right\},$$

from which they actually have started ($f(t)$ gives the intensity of renewal). Some formal connections with the integral equation of renewal theory are mentioned, but no results given.
W. Feller (Providence, R. I.).

***Tarjan, Rudolf. Untersuchungen über den Kapitalbedarf des Lebensversicherungsgeschäftes.** Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. III, pp. 335–351. Orell Füssli, Zurich, 1941. (French, Italian, English summaries)

The problem is reduced to the integral equation of self-

renewing aggregates in the form

$$u(t)=\phi(t)+\int_0^t u(x)p(t-x)dx,$$

where $\phi(t)$ and $p(t)$ are given. The solution is obtained in a formal way using the Laplace transform. The author expands the transform of $p(t)$ into a Taylor series and retains only a finite number of terms; this procedure allows an expansion into partial fractions of the transform of $u(t)$, but nothing is said about the degree of approximation.

W. Feller (Providence, R. I.).

Burkhardt, Felix. Über Stand und Wandlungen von bevölkerungs- und versicherungsstatistischen Personengesamtheiten. Bl. Versich.-Math. 5, 212–227 (1941). [MF 5437]

The author considers in an elementary fashion various functions such as mortality, reproductiveness, renewal, etc., of an aggregate which is heterogeneous with respect to age, sex, race or otherwise. *W. Feller* (Providence, R. I.).

Uggé, Albino. Sul metodo di eliminazione nella costruzione dei numeri indici dei prezzi. Pont. Acad. Sci. Acta 5, 67–71 (1941). [MF 5887]

Let p_{tk} be the price and q_{tk} the quantity consumed of the kth commodity ($k=1, \cdots, n$) in the time period t. Taking t as the basis period, Laspeyre's price index formula for the period t' is given by $P_{tt'}=\sum p_{t'k}q_{tk}/\sum p_{tk}q_{tk}$, and Paasche's formula is given by $P^*_{tt'}=\sum p_{t'k}q_{t'k}/\sum p_{tk}q_{t'k}$. The corresponding quantity indices $Q_{tt'}$ and $Q^*_{tt'}$ are obtained from the price indices by interchanging the prices with the quantities. The author shows that, if $P_{tt'}=P^*_{tt'}$, the following conditions of the method of elimination are fulfilled:

$$P_{tt'}=(P_{t't})^{-1}, \quad P_{tt'}Q_{tt'}=\sum_k p_{t'k}q_{t'k}/\sum_k p_{tk}q_{tk}.$$

A. Wald (New York, N. Y.).

GEOMETRY

MacNeish, H. F. Four finite geometries. Amer. Math. Monthly 49, 15–23 (1942). [MF 6141]

For each of four known finite plane geometries the author gives a system of axioms and proves that the axioms are independent. *G. de B. Robinson* (Ottawa, Ont.).

Artin, Emil. Coordinates in affine geometry. Rep. Math. Colloquium (2) 2, 15–20 (1940). [MF 5770]

The author considers two classes of elements, "points" and "lines," and assumes that (1) any two distinct points have a unique line joining them, (2) for each line l and point P not on l there is a unique line on P parallel to l, and (3) there are three points not on any line. A bi-uniform mapping which associates with each point P a point $P'=\sigma(P)$ is called a dilatation if $P\rightarrow P'$, $Q\rightarrow Q'$ implies the existence of two parallel lines l, l' with $P, Q\epsilon l$ and $P', Q'\epsilon l'$. If σ is a dilatation, any line containing a point P and its image P' is a trace of P. A translation is a dilatation without fixed points, or the identity. It is shown that the translations form a group which is Abelian in case there exist two translations with different traces. With each translation τ is associated a translation τ^α such that $(\tau_1\tau_2)^\alpha=\tau_1^\alpha\tau_2^\alpha$ and the traces of τ appear among those of τ^α. Such associations are called homomorphisms and are made the basis for introducing coordinates. Two additional axioms are needed to insure that the exponents form a field (with addition and multiplication properly defined), and that every translation τ is expressible uniquely as the product $\tau_1^\alpha\tau_2^\beta$, where τ_1 and τ_2 are two translations different from the identity and with different traces. Then the coordinates (α, β) of a point P are selected as the exponents of the homomorphisms τ_1^α, τ_2^β such that $\tau_{OP}=\tau_1^\alpha\tau_2^\beta$, where the point O is selected as the origin of the system, $\tau_1\neq 1$, $\tau_2\neq 1$ have different traces, and τ_{OP} is the unique translation sending O into P. Lines are represented by linear equations in these coordinates.

L. M. Blumenthal (Columbia, Mo.).

Cassina, U. Sul teorema fondamentale della geometria proiettiva ed i principii della geometria. Period. Mat. (4) 20, 65–83 (1940). [MF 5371]

This paper discusses briefly various postulational systems for projective geometry, including those of Severi, Pieri, and Peano. It contains a statement of the author's weakened form of the fundamental theorem of projective geometry [Ist. Lombardo, Rend. (3) 73 (1940)] from which it follows that a necessary and sufficient condition that a correspondence between two first order forms u and u' be a projectivity is that there exist three distinct elements a, b, c of u such that the correspondence preserves every harmonic quadruple containing a and every harmonic quadruple $(c, x; y, z)$ with $(a, b; y, z)=-1$. *L. M. Blumenthal.*

Barbilian, D. Zur Axiomatik der projektiven ebenen Ringgeometrien. II. Jber. Deutsch. Math. Verein. 51, 34–76 (1941).

Continuing his development of the foundations of projective "ring-geometry" in the plane [Jber. Deutsch. Math. Verein. 50, 179–229 (1940); these Rev. 2, 259], the author proves the independence of his system of axioms in § 4 and that the principle of duality is valid in § 5. Having previously defined the addition and multiplication of points, he investigates in § 6 the commutativity, associativity and distributivity of these operations and introduces a coordinate system in the usual manner. In conclusion he discusses the question of categoricalness. *G. de B. Robinson.*

Baer, Reinhold. Homogeneity of projective planes. Amer. J. Math. 64, 137–152 (1942). [MF 5994]

The validity of Desargues' theorem in the projective plane implies the possibility of setting up a coordinate system in that plane, while the validity of Pappus' theorem implies that the coordinate field is commutative. In this paper the author interprets these two theorems in terms of conditions which insure the existence of certain collineations and correlations. His condition implying the commutativity of the field in § 6 is particularly interesting. The symbolism may be made more symmetrical by interchanging $y(2)$ and $z(2)$; when this is done it is easy to see that $y(i)$, $z(i)$ and $L[p+x(i)]$, for $i=1, 2$, are point pairs of an involution on L, and that the condition is equivalent to the assumption that the involution is uniquely determined by $y(i)$ and $z(i)$.

G. de B. Robinson (Toronto, Ont.).

Kommerell, Karl. Die Pascalsche Konfiguration 9_3. Deutsche Math. 6, 16–32 (1941). [MF 5803]

In the figure for Pappus' theorem we have nine points lying by threes on nine lines: say points A, B, C, with

suffixes 0, 1, 2, such that A_i, B_j, C_k are collinear whenever $i+j+k\equiv 0 \pmod 3$. The group of automorphisms is of order 108; for, the letters A, B, C can be permuted in six ways, the suffixes of A in six ways, and those of B in three ways (cyclically). The author thoroughly investigates this group (without taking advantage of the fact that it occurs as a subgroup of index two in the well-known group of automorphisms of the nine inflections of a plane cubic curve).

The Pappus (or Pascal) configuration being self-dual, its nine lines can be named α, β, γ, with suffixes 0, 1, 2, in such a way that α_i, β_j, γ_k are concurrent whenever $i+j+k\equiv 0$ (mod 3). By changing A, B, C into α, β, γ, the author derives 108 dual automorphisms or reciprocities, of which 18 are involutory. It appears that none of the reciprocities are induced by correlations of the whole plane, while only three of the 108 ordinary automorphisms are induced by collineations. These three are the identity and $K^{\pm 1}$, where K is the cyclic permutation (012) applied to the suffixes of A, B, C (and α, β, γ) simultaneously. The last results might well have been checked by using projective coordinates of some such form as the following:

$$\begin{array}{lll} A_0(1, p, 1), & A_1(1, 0, 0), & A_2(0, p-q, p), \\ B_0(1, p, p), & B_1(0, 1, 0), & B_2(q-p, 0, q), \\ C_0(0, 0, 1), & C_1(1, q, 0), & C_2(1, 1, 1). \end{array}$$

H. S. M. Coxeter (Toronto, Ont.).

Haenzel, G. Die Diracsche Wellengleichung und das Ikosaeder. J. Reine Angew. Math. **183**, 232–242 (1941). [MF 5725]

In two previous papers [Jber. Deutsch. Math. Verein. **49**, 215–242 (1940); **50**, 121–129 (1940); cf. these Rev. **1**, 262; **2**, 152] the author has given representations of Eddington's operators E_{ik} as applied to Dirac's wave equation in terms of the Pascal and Brianchon configurations. On account of the close relationship of these figures with the icosahedron he is able to carry over his representation and to interpret the operators in terms of this regular polyhedron. (A similar argument would also apply to the dodecahedron.) The two electron problem may be discussed in terms of two icosahedra which are transformable into one another by rotating through an angle of $\pi/2$.

G. de B. Robinson (Toronto, Ont.).

Carver, W. B. The polygonal regions into which a plane is divided by n straight lines. Amer. Math. Monthly **48**, 667–675 (1941). [MF 6026]

A set of n lines, no three concurrent, forms a figure F_n which separates the projective plane into polygonal regions. Each of the α_m m-sided regions R_m is projectively equivalent to a convex m-sided polygon whose sides are segments of m of the given lines. For $m\geqq 5$ the convexity of R_m prescribes a certain reversible cyclic order to the m lines, namely, the order in which they bound the region R_m. Three or four lines are said to be convex in any order, but five lines are convex in only one reversible cyclic order, and more than five lines in at most one such order. Two figures F_n and F_n' are said to be equivalent if a one to one correspondence exists between the lines of F_n and those of F_n' such that, whenever m ordered lines of F_n bound a region R_m of F_n, the corresponding ordered lines of F_n' bound a region R_m' of F_n'. The problem of determining the nonequivalent figures F_n for $n\leqq 8$ was tackled in an observational manner by H. S. White [Bull. Amer. Math. Soc. **38**, 59–65 (1932)] and L. D. Cummings [Bull. Amer. Math. Soc. **38**, 105–110, 700–702 (1932); **39**, 411–416 (1933)]. The two Diophantine equations $\sum\alpha_m=(n^2-n+2)/2$, $\sum m\alpha_m=2(n^2-n)$ are necessary but not sufficient conditions for a solution, and it may happen that two non-equivalent figures F_n have the same α_m. The author places the problem partly on an analytic basis by defining a five line function $[i\cdot jk\cdot lm]$ which is $+1$ or -1 according as the points in which line p_i meets p_j, p_k, p_l, p_m have a positive or negative cross ratio, and showing that the conditions $[1\cdot 24\cdot 35]=[4\cdot 52\cdot 13]=-1$ are necessary and sufficient for the convexity of the ordered set 1, 2, 3, 4, 5. He shows that every F_n with $n\geqq 5$ contains an R_m with $m\geqq 5$, and that there is just one F_n with $\alpha_n=1$, and this has $\alpha_3=n$, $\alpha_4=(n^2-3n)/2$. Another type of F_n for even n is described in which $\alpha_{n-1}=1$. Finally a theorem of N. G. Gunderson is quoted: "If in a figure F_n there exist two different regions R_m and R_s, then $m+s\leqq n+4$." One type of F_8 not listed in the Cummings table given in the paper, but readily obtained by adding an extra line to Fig. 2, has $\alpha_3=12$, $\alpha_4=10$, $\alpha_5=4$, $\alpha_6=2$.

J. S. Frame (Providence, R. I.).

Schilling, Friedrich. Die Extremaleigenschaften der ausserhalb des absoluten Kegelschnittes gelegenen Strecken in der projektiven Ebene mit hyperbolischer Geometrie. Deutsche Math. **6**, 33–49 (1941). [MF 5804]

The author considers the differential geometry of the ultra-infinite region of the hyperbolic plane, that is, of the points exterior to the absolute. He shows that this region contains triangles ABC with $AC+CB<AB$, one type in which the sides are segments of ordinary lines (produced), and another in which they are segments of ultra-infinite lines. (In the latter case the distances become real when we divide them by i, the metric in an ultra-infinite line being elliptic.) He deduces that a segment of either type is, in a well defined sense, the longest path joining its end points. (The alternative paths must be curves whose tangents vary continuously without ever becoming isotropic.) [Considerable labor could have been saved by using the methods of pure hyperbolic geometry with projective coordinates, instead of a Euclidean model with Cartesian coordinates.]

H. S. M. Coxeter (Toronto, Ont.).

Jenks, Frederick P. A new set of postulates for Bolyai-Lobachevsky geometry. II. Rep. Math. Colloquium (2) **2**, 10–14 (1940). [MF 5769]

Jenks, Frederick P. A new set of postulates for Bolyai-Lobachevsky geometry. III. Rep. Math. Colloquium (2) **3**, 3–12 (1941). [MF 5776]

It is known that all the concepts of the Bolyai-Lobachevsky geometry can be defined in terms of the operations "joining" and "intersecting." The purpose of these papers is to show how the theory of order and parallelism for the plane Bolyai-Lobachevsky geometry can be based on a few simple postulates concerning the two lattice operations. An abstract of the theory is contained in an earlier paper [Proc. Nat. Acad. Sci. U. S. A. **26**, 277–279 (1940); these Rev. **1**, 261]. In part I the author defined the idea of betweenness in terms of join and intersection and formulated seven postulates, involving only these concepts, from which he could derive the properties of betweenness. In part II he proves, after having formulated an eighth postulate, the theorem of Pasch and a few related theorems. Part III deals with the theory of parallelism, which is based on the following definition: two lines a and b are said to be parallel if they do not intersect and there exists a point between a and

b (the definition of this betweenness has been given already) such that through P there is at most one line which meets neither a nor b. The eight axioms formulated so far permit the statement that, if a is a line and P is any point not on a, then through P there exist infinitely many lines which do not intersect, and also the statement that there exist at most two parallels to a line through a point not on the line. In order to prove that at least two parallels through P exist, a ninth postulate must be formulated. *E. Helly.*

Abbott, James C. The projective theory of non-Euclidean geometry. I. Rep. Math. Colloquium (2) **3**, 13–27 (1941). [MF 5777]

F. P. Jenks developed in his papers reviewed above the theory of order and parallelism for the non-Euclidean plane using only the operations of joining and intersecting and nine postulates concerning them. The author of this paper intends to develop the theory of congruency and perpendicularity, based on Jenks' postulates and two additional assumptions, and to prove that the geometry obtained in this way is identical with the classical theory. Section I of the present paper extends the plane of Jenks to a closed plane by defining ends or infinite points as pairs of parallel lines. Section II introduces the fundamental theorem of projective geometry as a new postulate and defines projectively the notion of congruency of two segments. The theory is to be continued and completed in a following paper. *E. Helly* (Paterson, N. J.).

Menger, Karl. On algebra of geometry and recent progress in non-Euclidean geometry. Rice Inst. Pamphlet **27**, 41–79 (1940). [MF 6081]

In this second of "Three Lectures on Mathematical Subjects" delivered at the Rice Institute in 1939, the author reviews recent contributions to the axiomatics of projective, affine, Euclidean and hyperbolic geometry given in a series of publications by himself and his students. The author has developed projective geometry in terms of a single class of undefined entities and two primitive notions "projection" and "section," denoted by $+$ and $\cdot$, respectively. The postulates are statements concerning the results of "adding" and "multiplying" the entities, and their algebraic appearance suggested the term "algebra of geometry." [See Menger, Jber. Deutsch. Math. Verein. **37**, 309–325 (1928); Ann. of Math. (2) **37**, 456–482 (1936), the latter written in collaboration with F. Alt and O. Schreiber.] The system is remarkable in several ways: for example, (1) it starts with a single class of entities and later classifies them according to a dimension concept based upon the relations of the entities with respect to the two fundamental operations, thus permitting the development of projective geometry for spaces of any finite number of dimensions without the necessity of introducing new postulates and definitions; (2) it permits the simultaneous development of projective and affine geometry by first exploiting the consequences of that part of the postulates valid in both geometries. Adjoining the dual of one of these postulates gives a self-dual system on which projective geometry is based, while adding the Euclidean parallel postulate yields affine geometry. The question of basing other geometries upon the two operations of joining and intersecting naturally arises. As a consequence of a theorem of Padoa, it may be shown that Euclidean geometry cannot be founded upon the notions of joining and intersecting since congruency and perpendicularity are not definable in terms of these notions. In view of this fact, it is highly interesting that the author has shown that hyperbolic geometry (with its notions of perpendicularity and congruence) can be based upon the sole operations of joining and intersecting [Bull. Amer. Math. Soc. **44**, 821–824 (1938); C. R. Acad. Sci. Paris **207**, 458–460 (1938); Proc. Nat. Acad. Sci. U. S. A. **24**, 486–490 (1938)]. The lecture closes with a brief discussion of the developments of hyperbolic geometry in terms of joining and intersecting given by F. P. Jenks [Proc. Nat. Acad. Sci. U. S. A. **26**, 277–279 (1940); these Rev. **1**, 261] and J. C. Abbott in their Notre Dame dissertations. The definitions of betweenness and parallelism contained in Jenks' paper improve upon the definitions given by Menger. In the work of Abbott the theorem of Pappus plays a central role, for upon it Abbott bases his treatment of perpendicularity and congruence. *L. M. Blumenthal* (Columbia, Mo.).

Petkantschin, Bojan. Über die Orientierung der Kugel in der Möbiusschen Geometrie. Jber. Deutsch. Math. Verein. **51**, 124–147 (1941).

The author gives an axiomatic treatment of two-dimensional Möbius geometry in terms of point, circle, incidence of point and circle, and ten axioms of existence, connection and order. Oriented circular arcs, circles and hemispheres are defined, the last leading to the notion of orientation of the sphere. *L. M. Blumenthal* (Columbia, Mo.).

Busemann, Herbert. Metric conditions for symmetric Finsler spaces. Proc. Nat. Acad. Sci. U. S. A. **27**, 533–535 (1941). [MF 5733]

With the object of characterizing metrically the class of symmetric Finsler spaces, the author investigates finitely compact, convex, metric spaces satisfying two conditions D and Δ, which are properties of a Finsler space whose metric is determined by a function $F(x, \lambda)$ of contravariant vectors (x, λ) which satisfies in addition to the usual conditions the relation $F(x, \lambda) = F(x, -\lambda)$. Condition D is that of strict external convexity in the small, and has as easy consequences (1) the uniqueness of segments and their unique prolongation in the small, (2) imbedding of a segment in a unique geodesic, and (3) in case the space is one- or two-dimensional it is a manifold. The reader is referred to the paper for the rather complicated condition Δ. With its use the space may be proved to be a coordinate manifold of finite dimension (that is, each point has a neighborhood homeomorphic with the interior of the n-dimensional sphere) and each point has a neighborhood in which a Minkowskian metric may be defined which is equivalent to the original metric. A more extended review of this interesting paper might well await the appearance of the detailed proofs which are promised in a forthcoming monograph. *L. M. Blumenthal* (Columbia, Mo.).

Pepper, Paul M. Concerning pseudo planar quintuples. Rep. Math. Colloquium (2) **2**, 28–32 (1940). [MF 5772]

A metric quintuple is pseudo-planar provided each of its quadruples is congruently contained in the plane, while the quintuple is not. Such quintuples are not congruently imbeddable in any Euclidean space and the problem arises of determining spaces as "nearly Euclidean" as possible in which they may be so imbedded. The author presents two such spaces. One is the three-leaf which is formed by metrizing convexly the union of three half-planes with a common axis, and the other consists of a triangular dihedron made by joining two congruent triangles along their equal

edges and defining the metric pq as Euclidean for points p, q in the same face and $pq = \min (pr+rq)$, $r\epsilon$ boundary, for p, q in different faces. The three-leaf contains congruently all pseudo-planar quintuples, while each member of the three-parameter family of triangular dihedra contains congruently a two-parameter family of pseudo-planar quintuples. *L. M. Blumenthal* (Columbia, Mo.).

Pepper, Paul M. Nearly Euclidean imbedding spaces for pseudo-E_n sets. Rep. Math. Colloquium (2) 3, 34–46 (1941). [MF 5780]

This paper generalizes to pseudo-E_n $(n+3)$-tuples the results obtained in the paper reviewed above for pseudo-planar quintuples. If E_n', E_n'' are two n-dimensional Euclidean spaces, S', S'' congruent sets of $n+1$ independent points of E_n', E_n'', respectively, and H', H'' the union of the hyperplanes of the faces of the respective simplices determined by S', S'', the space D, whose points are those of E_n' and E_n'', is metrized by attaching the Euclidean metric to each pair of points both of which belong to E_n' or to E_n'', while for $x\epsilon E_n'$, $y\epsilon E_n''$ the distance xy is defined by $xy = \text{g.l.b.}_{h'\epsilon H'}[xh' + g(h')y]$, where g is a congruent mapping of H' onto H''. Each metric $(n+1)$-tuple S which is congruent with $n+1$ independent points of an n-dimensional Euclidean space "generates" such a space $D(S)$. It is shown that $D(S)$ is convex, locally externally convex, complete, finitely compact and metric and each pseudo-E_n set P contains at least one $(n+1)$-tuple S which generates a $D(S)$ into which the pseudo-E_n set may be congruently imbedded. A universal imbedding space for all pseudo-E_n sets, n fixed, is constructed in a similar manner. It may be remarked that Lemmas 2.1 and 2.2 of the paper are well known. *L. M. Blumenthal* (Columbia, Mo.).

Algebraic Geometry

Terracini, Alejandro. On some geometrical loci. Revista Union Mat. Argentina 7, 97–105 (1941). (Spanish) [MF 6180]

Given a triangle $A'B'P'$ and two analytic curves a, b, all in the same plane. Associate with a variable point A on a and a variable point B on b a definite point P such that the triangle ABP shall be directly similar to the given $A'B'P'$. Under what conditions can the locus p of P be a curve, rather than a region of the plane? In the present paper five solutions are given, in each of which p is a straight line. These are: (1) a, b are two non-parallel lines, and p is a line of their pencil; (2) a an isotropic line and p coincides with it; (3) a limiting case of (2); (4) a limiting case of (1), in which a and b are parallel and the points A', B', P' are collinear; (5) a projective generalization of (2). The author then proves that these cases comprise all possible ones. *V. Snyder* (Ithaca, N. Y.).

Campedelli, Luigi. Una costruzione proiettiva delle trasformazioni piane del De Jonquières. Boll. Un. Mat. Ital. (2) 3, 102–108 (1940). [MF 5579]

A birational transformation between two planes π and π' is constructed as follows. Given in π a pencil of rational, irreducible curves, each of order $n-1$ with an $(n-2)$-fold basis point at O and $2n-3$ simple basis points, projective with a pencil of lines on a fixed point O' of π'; and also a pencil of lines on O in π projective with a pencil of lines on a fixed point A' of π'. A point P' of π', defined by one ray each of the pencils on O' and A', has its image P in π at the intersection of the corresponding curve and line in the respective projectivities described above. This is a de Jonquières transformation of order n. The characteristics of the transformation are found and particular cases discussed. The author states that, although this construction "is similar in a certain sense to the spirit of the original study by de Jonquières," he has not found any mention of it after a sufficiently wide search. *T. R. Hollcroft.*

Cherubino, Salvatore. Sulle corrispondenze algebriche fra curve. Ann. Scuola Norm. Super. Pisa (2) 10, 1–11 (1941). [MF 5469]

This paper is divided into three sections. In the first section it is shown that certain relations from which R. Torelli hoped to obtain relations among the normal periods of a fundamental system of integrals associated with an algebraic curve are reducible to those previously obtained by Riemann. In section two algebraic correspondences between two curves of the same genus are characterized by products of the matrices of the correspondence and its inverse. In section three necessary and sufficient conditions are obtained that two curves with birationally equivalent Jacobians be birationally equivalent. *T. R. Hollcroft.*

Corradi, Maria Virginia. Sulle singolarità della curva Hessiana. Boll. Un. Mat. Ital. (2) 3, 215–218 (1941). [MF 5591]

More than a half century ago, Del Pezzo found that a necessary condition for the Hessian H of an algebraic curve F to have a node at a non-singular point O of F is that O be an undulation of F; and that this condition is sufficient if and only if "some of the coefficients of F satisfy a relation that does not appear to have a simple geometric interpretation." If O is an undulation on F, the tangent t to F at O is a triple tangent with three coincident contacts. The polar cubic of O with respect to F breaks up into t and a conic γ not through O. The author finds that the additional geometric condition that H have a node at O is that t be tangent to γ. *T. R. Hollcroft* (Aurora, N. Y.).

Buzano, Piero. Sull'invariante proiettivo di una terna di elementi curvilinei del 1° ordine. Boll. Un. Mat. Ital. (2) 3, 201–207 (1941). [MF 5587]

Three line elements $(ux)=(vy)=(wz)=0$ have a unique projective invariant $I=(vx)(wy)(uz)/(wx)(uy)(vz)$. If $I=1$ the three elements are on a conic. The author shows that a curve, any three of whose line elements have the same invariant, is necessarily a conic. Two further theorems are proved. (1) The elements which have with two fixed elements a constant invariant lie on a family of W-curves. (2) Given two fixed elements and an algebraic curve C^n. For any constant q let S_q be the set of points of C such that for the tangent line element and the two given elements we have $I=q$. Then S_q is a set of a linear series g_r^1 on C^n. The value of r depends on the position of the given elements, and is easily determined. *R. J. Walker.*

Coble, A. B. Trilinear forms. Duke Math. J. 7, 380–395 (1940). [MF 3409]

This paper is concerned with the trilinear form

$$T(m, n, p) = (\alpha x)(\beta y)(\gamma z) = 0,$$

in which x, y, z are points in $[m]$, $[n]$, $[p]$, respectively, with prime coordinates ξ, η, ζ and $1 \leqq m \leqq n \leqq p$. It is as-

sumed that $T=0$ has no neutral points, that is, points for which the bilinear form in the remaining variables vanishes identically. But neutral pairs do exist. Thus, x and y are a neutral pair for z if their coordinates satisfy the equations $(\alpha x)(\beta y)\gamma_k=0$ for $k=0, 1, \cdots, p$, and similarly for the other pairs. The associated matrices are studied extensively under the notation $V(|m+1, n+1|, [p])$ by G. T. Room [Geometry of Determinantal Loci, Cambridge, England, 1938]; V is of dimension $p+m-n-1$ and of order $\binom{n+1}{m}$ in $[p]$. When $p+1=m+n$ a finite number $N=\binom{m+n}{m}$ of neutral pairs x, y exist. The points x comprise a set P_N^m in $[m]$ and their associates y a set Q_N^n in $[n]$. These sets are ordered. When x, y are a neutral pair of points in T, the generators of V are spaces $[p-n]=\pi$ and $[p-m]=k$; the spaces π lie on V.

The chief interest in the paper is when $m=2$; its purpose is to determine T in terms of a generic ternary set of points P. The bordered determinant

$$\begin{vmatrix} (\alpha x)\beta_\zeta\gamma_k \\ \zeta_k \end{vmatrix}=0$$

for given ζ defines a curve $Q^{n+1}(\zeta)$ of a ∞^{n+1} linear system of curves of order $n+1$ in the plane x on the set P. This determinant equation is the map of the plane $[x]$ on the points of a White surface [Room, loc. cit., chap. 14] in $[z]$. The lines π and the primes k form a double N. It is shown that $T(2, n, n+1)$ with N pairs x, y neutral for z is projectively defined by a generic set P of points x of these pairs. A prime section of a White surface $F(\zeta)$ in $[z]$ corresponds to a curve $Q^{n+1}(\zeta):P$. There are ∞ sections $[n-2]$ which cut F in $Nn-2$ points all lying on a curve. Among them are the Semple congruence of $[n-2]$'s [Room, loc. cit., p. 383], which are studied in detail. *V. Snyder.*

Coble, Arthur B. Conditions on the nodes of a rational plane curve. Duke Math. J. 7, 396–410 (1940). [MF 3410]

The rational plane curve of order $n+3$ has

$$N=(n+1)(n+2)/2$$

nodes at points P. For $n<3$, the nodes may be taken arbitrarily; for $n=3$, the ten nodes are subject to three conditions [A. B. Coble, Amer. J. Math. **41**, 243–265 (1919); in particular, pp. 251–254]. It is the purpose of this paper to obtain necessary conditions on the set P for generic n, not hitherto known. A set N of "nodular" points P' is defined, in general not projective with P, and a set Q of "catalectic" points appears in space $[n]$ which defines a rational envelope associated with the given curve. A trilinear form $T'(2, n, n+1)$ in x', y, z with neutral pairs x, y for z at the sets P', Q, respectively, and another trilinear form $T(2, n, n+1)$ in x, y, z with neutral pairs of x, y for z at points of the sets P, Q, respectively [cf. the preceding review], are found. The identity of Q for the non-projective sets P, P' yields the conditions desired. It all follows from an appropriate coordinate and parameter system.

V. Snyder (Ithaca, N. Y.).

Leidheuser, R. W. Über die reine Darstellung algebraischer Raumkurven. Ein Brief an den Herausgeber. Deutsche Math. 6, 1–2 (1941). [MF 5801]

A recent paper [O. Perron, Math. Z. **47**, 319 (1941)] contends that the theorem that a space curve may require four surfaces for its pure representation is not established. The present note points out that the contention is due to a misconception of the word pure. The theorem still holds [L. Kronecker, J. Reine Angew. Math. **92**, 1–122 (1882) with an illustration confirming the theorem of K. Th. Vahlen, J. Reine Angew. Math. **108**, 346–347 (1891)].

V. Snyder (Ithaca, N. Y.).

Franchetta, Alfredo. Sulle curve eccezionali di prima specie appartenenti ad una superficie algebrica. Boll. Un. Mat. Ital. (2) 3, 28–29 (1940). [MF 5572]

An irrational algebraic surface has only a finite number of exceptional curves of the first kind (those eliminable by birational transformation without introducing new ones). To rational surfaces the usual proofs do not apply. The present paper constructs rational surfaces with an infinite number of exceptional curves of the first kind.

V. Snyder (Ithaca, N. Y.).

Franchetta, A. Sulla caratterizzazione delle curve eccezionali riducibili di prima specie. Boll. Un. Mat. Ital. (2) 3, 372–375 (1941). [MF 5705]

This paper proves the following theorem. Given a virtually connected curve belonging to an algebraic surface not reducible to a ruled surface; if the curve is of virtual genus zero and of virtual grade -1, it is an exceptional curve of the first kind. That is, a birational transformation can be found which transforms this curve into a point without transforming any point of the surface into a curve [Barber and Zariski, Amer. J. Math. **57**, 119–141 (1935)].

V. Snyder (Ithaca, N. Y.).

Anand, Jagdish Gopal. On apolar quadrics. Bull. Calcutta Math. Soc. 33, 45–47 (1941). [MF 6168]

The chief purpose of this paper is to prove by using only the properties of the following figures that, given three types of self-conjugate figures, a self-polar tetrahedron t, a self-conjugate pentad p and a self-conjugate hexad h all with respect to a quadric Σ, if the vertices of any one of the three types t, p, h lie on a quadric S, then S contains the vertices of an infinite number of each of the other two types. A proof of this given previously by Miss Telling [Proc. Cambridge Philos. Soc. **21** (1923)] depends chiefly on the methods of construction of the figures. The following property of a cubic curve σ outpolar to Σ is also proved. If on this cubic σ occur the vertices of either p or h, then σ circumscribes an infinite number of figures of type t.

T. R. Hollcroft (Aurora, N. Y.).

Mandan, Ram. A relation between a pencil and a range of quadrics. Proc. Indian Acad. Sci., Sect. A. 14, 313–316 (1941). [MF 6070]

The author establishes a geometric relation between quadric surfaces and uses it to prove the equality of certain complicated algebraic relations. *R. J. Walker.*

Pedoe, D. A remark on a property of a special pencil of quadrics. Proc. Cambridge Philos. Soc. 38, 220–221 (1942). [MF 6254]

The following theorem is proved. Let Q be a non-singular quadric hypersurface and L a hyperplane not tangent to Q. If T_i and M_i denote the total and the mean curvatures of $Q+\lambda_i L^2$ at a point P common to Q and L, then the cross ratios $\{T_1T_2T_3T_4\}$, $\{M_1M_2M_3M_4\}$, $\{\lambda_1\lambda_2\lambda_3\lambda_4\}$ are equal.

R. J. Walker (Ithaca, N. Y.).

Walden, Earl. Algebraic surfaces invariant under the symmetric G_{120} with special reference to quintics and sextics. Nat. Math. Mag. 15, 163–172 (1941). [MF 5688]

The author investigates the surfaces which are invariant under the symmetric group G_{120} on five variables x_i and which lie in the hyperplane $\varphi_1 \equiv x_1+x_2+x_3+x_4+x_5=0$ of projective four-space. The five planes π_i $(x_i=0)$ form a pentalateral whose five faces, ten edges and ten vertices, together with certain diagonal points, diagonal lines and diagonal planes, and the polar reciprocals of these with respect to the quadric $\sum x_i^2=0$, define the most important sets of points, lines and planes invariant under G_{120}. The equation of the general surface F_n of order n invariant under G_{120} is expressible as a polynomial in the symmetric functions φ_2, φ_3, φ_4, φ_5 of the x_i's. Several theorems of the paper state conditions on the order n, such that various sets of points or lines should be invariant. A study is then made of invariant curves, quintic surfaces and sextic surfaces. Certain particular quintics and sextics which have double points or lines on them are singled out for special comment. *J. S. Frame* (Providence, R. I.).

Enriques, Federico. Sur l'extension du théorème de Riemann-Roch aux systèmes linéaires de courbes appartenant à une surface algébrique. Bull. Sci. Math. (2) 64, 207–215 (1940). [MF 5234]

The Riemann-Roch theorem defining the dimension of a complete linear series of point groups on an algebraic curve was extended in 1886 by M. Noether to linear curve systems on surfaces. Noether obtained the equality $r=p+n-\pi+1-i$, wherein r is the dimension of the linear curve system $|C|$ of order n and genus π on an algebraic surface F of genus p, and i is the index of specialty of $|C|$. Almost a half century ago, Enriques and Castelnuovo showed that this is an equality usually only when F is regular (that is, $p_g=p_a$, where p_g and p_a are the geometric and arithmetic genera, respectively, of F) and that for F irregular, that is, $p_g>p_a$, it becomes $r\geqq p_a+n-\pi+1-i$. In the present paper, the author shows that the defect of r, that is, the difference between this lower limit and the value of r for a given F, does not exceed p_g-p_a. Finally, it is shown that, given a linear irreducible system $|C|$ of curves of genus π on an irregular surface F of arithmetic genus p_a, there exists a linear system $|C'|$ adjoint to $|C|$ whose dimension has the exact value $i-1=p_a+\pi-1$, where i is the index of specialty of the series cut out on a curve of the system $|C+D|$ by the system $|D|$, $|D|$ being also a linear irreducible system on F. This theorem was proved in 1905 by E. Picard using transcendental methods, and is established in the present paper by elementary methods, with the restriction that the dimension of $|C|$ be greater than p_g-p_a. *T. R. Hollcroft* (Aurora, N. Y.).

Godeaux, Lucien. Remarques sur les surfaces multiples ayant un nombre fini de points de diramation. Bull. Sci. Math. (2) 64, 245–256 (1940). [MF 5237]

An algebraic surface F, normal in S_r, has on it a cyclic involution I_p, determined by a homography of period p and possessing a finite number of coincidence points A_i. In a projective transformation of S_r, F corresponds to a normal surface Φ on which occur branch points A_i' in (1, 1) correspondence with the coincidence points A_i. Assuming that the branch points A_i' are non-singular or are various types of simple and compound double points on Φ, the author obtains the corresponding properties of the involution, including the nature of the coincidences at the A_i, the multiplicities of the prime sections $|C|$ of F at the A_i and the properties of the curves on Φ which are images of $|C|$. *T. R. Hollcroft* (Aurora, N. Y.).

Godeaux, Lucien. Sur la structure des points unis des homographies cycliques du plan. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. 19, fasc. 1, 42 pp. (1941). [MF 5833]

The author has contributed many papers to the study of involutions with a finite number of fixed points belonging to an algebraic surface [mostly in Mém. Soc. Roy. Sci. Liége and Acad. Roy. Belgique. Bull. Cl. Sci. from 1928 to 1940], involving the properties of non-perspective plane homologies H of prime period $p>2$. Given $|C|$ a linear system of plane curves of order p, without base points, invariant under H. Two such curves meet in p^2 points, forming two groups of an involution I_p. The curves of the system are mapped on the primes of a projective space S with dimension equal to that of $|C|$, forming a normal surface Φ of order p, image of I_p; H has three distinct fixed points A_i. Those curves C' of $|C|$ through A_1 have two fixed tangents A_1A_2 and A_1A_3. From the equations of H it follows that the curves C' have a series of adjacent multiple points at each point A_i, each representing one or more groups of I_p, the end-point of which is a perfect point. Some of these properties had been obtained in earlier papers by the same author [l.c.]. *V. Snyder* (Ithaca, N. Y.).

Godeaux, Lucien. Sur les surfaces du quatrième ordre possédant quatre points doubles uniplanaires. Mathematica, Cluj 17, 19–29 (1941). [MF 5693]

In earlier papers the same author considered a quartic surface in [3] having twelve double conical points [Acad. Roy. Belgique. Bull. Cl. Sci. (5) 6, 554–560 (1920)] in which the surface is expressed in a pencil determined by the four faces of a tetrahedron and a general quadric counted twice. A particular case [ibid. (5) 10, 418–428 (1924)] is given when the quadric F passes through the vertices of the tetrahedron, which are now biplanar double points. In the present paper a surface of order 8 in [5] is constructed invariant under a non-cyclic group of linear transformations of order four. The involution on the surface belongs to F. Finally it is shown that the involution can be reduced to one having simple points of the supporting surface for fixed points. *V. Snyder* (Ithaca, N. Y.).

Edge, W. L. Some remarks occasioned by the geometry of the Veronese surface. Proc. Roy. Soc. Edinburgh. Sect. A. 61, 140–159 (1941). [MF 5977]

The paper contains an expository discussion of the Veronese surface [G. Veronese, Mem. Accad. Naz. Lincei (3) 19, 344–371 (1884)], combining a geometric representation with a new algebraic treatment that brings out correctly the self-dual properties of the surface. Various properties of the secant planes (those meeting the surface in conics) are added, including their relations to quadric primals which either contain the surface or are outpolar to it. A new feature of the paper is the derivation of an invariant and of two contravariants of a ternary quartic by means of the (1, 1) correspondence between a linear system of plane quartic curves and these outpolar quadric primals. The treatment is first geometric, followed by a

purely algebraic proof of the same theorems. Various errors and unsymmetric discussions in the literature [J. J. Sylvester, Collected Mathematical Papers, Cambridge, England, 1904, vol. 1, pp. 61–65] are replaced, in the present paper, by a consistent and completely self-dual presentation.

V. Snyder (Ithaca, N. Y.).

Differential Geometry

Kasner, Edward and De Cicco, John. Infinite groups generated by equilong transformations of period two. Amer. J. Math. **63**, 709–725 (1941). [MF 5619]

In their study of equilong transformations the authors are guided by the same program that Kasner followed in his 1916 paper [Amer. J. Math. **38**] devoted to conformal transformations. They had already found [Proc. Nat. Acad. Sci. U. S. A. **26**, 287–291, 471–476 (1940); these Rev. **2**, 15] that there are three types of equilong transformations of period two. They first study the groups generated by transformations of each type separately. Every transformation of the group generated by (equilong) involutions may be presented as a product of four or fewer involutions, and the group generated by inversions has a similar structure. On the other hand, the group generated by symmetries (which, incidentally, are given here a simple geometrical interpretation) consist only of symmetries and equilong translations which are defined as products of two symmetries. Finally, the group generated by all equilong transformations of period two is considered; it may also be generated by inversions and symmetries alone; its transformations may be written as products of four or fewer inversions and one or no symmetry. *G. Y. Rainich.*

Feld, J. M. The geometry of whirls and whirl-motions in space. Bull. Amer. Math. Soc. **47**, 927–933 (1941). [MF 5940]

The author develops in three-space S_3 a theory analogous to the plane geometry of whirls and whirl-motions of Kasner [Amer. J. Math. **33**, 193–202 (1911)]. The basic element of this theory consists of a point of S_3 associated with an ordered pair of directions at this point. Such plane elements are given coordinates by the use of quaternions, and various transformations (slides, turns and whirls) are defined and shown to form groups. An analogue of Kasner's turbine is obtained and shown to be invariant under a certain group of whirl-motions. Finally, the relation of these objects to certain kinematical concepts is indicated. *R. J. Walker.*

Cohn, Richard. Some exceptional values of the limit of the ratio of arc to chord. Bull. Amer. Math. Soc. **47**, 746–749 (1941). [MF 5479]

The author studies the limiting value of the ratio of the arc of an analytic curve to its chord AB when A is fixed and B approaches A. He proves results for the Kasner plane $(ds=dx/y')$ and for the Blaschke plane $(ds=[y'']^{1/3}dx)$ analogous to the theorem of Kasner for the complex Euclidean plane, namely: The limit is unity for all curves not tangent to a minimal curve at A. For curves having contact of the nth order with a minimal curve, the limit is a function of n. The function of n is different for the different geometries, but in each case is unity for the lowest possible value of n, and not unity for the other cases. *P. Franklin.*

Humbert, Pierre. Sur certaines figures planes de l'espace attaché à l'opérateur Δ_3. C. R. Acad. Sci. Paris **211**, 530–531 (1940). [MF 5360]

Distance of a point in the plane from the origin being defined by $(x^3+y^3)^{\frac{1}{3}}$, the author finds that the "altitudes of a triangle" meet in one point. Reference is made to a previous paper [C. R. Acad. Sci. Paris **209**, 590–591 (1939); cf. these Rev. **1**, 172] and to a paper appearing in J. Math. Pures Appl. *O. Szász* (Cincinnati, Ohio).

Goormaghtigh, R. Solution géométrique des problèmes fondamentaux de la géométrie intrinsèque plane. Mathesis **54**, 178–182 (1940). [MF 5161]

Let φ be the angle of contingence at a point M of the curve (M) generated by M. Let the local coordinates of the points P, Q be functions of φ. Denote by P', Q' the points whose local coordinates are the derivatives with respect to φ of the local coordinates of P, Q, respectively. These ideas are used in determining intrinsic geometrical properties of the loci of the points M, P, Q and of the envelopes of the lines joining pairs of these points. Typical theorems are: The normal at P to the curve (P) described by P passes through P'. The point of contact of the line PQ with its envelope is the center of similitude of the segment PQ and the segment determined by the projection of $P'Q'$ on PQ. In particular the line through P making a constant angle with the tangent to (M) touches its envelope at the projection of P' on this line. *V. G. Grove.*

Egerváry, Eugène et Alexits, Georges. Fondements d'une théorie générale de la courbure linéaire. Comment. Math. Helv. **13**, 257–276 (1941).

In a semimetric space E for any n points p_i let

$$D(p_1, \cdots, p_n)=\begin{vmatrix} 0 & 1 \\ 1 & (p_ip_j)^2 \end{vmatrix}_{(i,\, j=1,\, \cdots,\, n)}$$

and for any $n+2$ points $p_0, p_1, \cdots, p_{n+1}$ let

$$\kappa(p_0, p_1, \cdots, p_{n+1})=\frac{n+1}{p_0p_{n+1}}\,|D(p_0, \cdots, p_{n+1})D(p_1, \cdots, p_n)$$
$$\times D(p_0, \cdots, p_n)^{-1}D(p_1, \cdots, p_{n+1})^{-1}|.$$

If $\kappa(p_0)=\lim \kappa(p_0, \cdots, p_{n+1})$ exists for $p_i\to p_0$, $i=1, \cdots, n+1$, it is called the nth linear curvature of E at p_0. It is shown that for sufficiently smooth curves in a Euclidean space E_k this definition coincides with the usual nth curvature, namely, the rate with which the osculating L_n changes with respect to the arc length s. If in any semimetric space instead of $\kappa(p_0)$ the stronger limit $\kappa^*(p_0)=\lim \kappa(p_1, \cdots, p_{n+2})$ exists for $p_i\to p_0$ then $\kappa^*(p_0)=\kappa(p_0)$ is continuous. Conversely, if $\kappa(p_0)$ exists and is continuous, then $\kappa^*(p_0)$ exists. In that case $\kappa(p_0)$ is approached uniformly by $\kappa(p_0, \cdots, p_{n+1})$. The implications of the existence of $\kappa(p_0)$ for continua E in E_k are discussed and results like the following are obtained: If $\kappa(p_0)$ exists at p_0 and is finite, then E has at p_0 an osculating L_n. If for a curve $x_i=x_i(s)$, $i=1, \cdots, k$, the $x_i(s)$ have $n+1$ derivatives in the neighborhood of s_0, which are continuous at s_0, then $\kappa^*(x(s_0))$ exists.

H. Busemann (Chicago, Ill.).

Sbrana, Francesco. Una generalizzazione della normal affine ad una curva piana. Boll. Un. Mat. Ital. (2) **3**, 210–212 (1941). [MF 5589]

Given a curve C with an ordinary tangent t at a point O. From any point Q of t, lines cut the branch of C near O in

two points P_1 and P_2. The point N divides the segment P_1P_2 in the ratio $1:k$. The locus of N is a curve Γ which, for $k\neq 1$, has simple contact with C at O, the invariant of contact being $(k+1)^2/(k-1)^2$. For $k=1$, Γ intersects C at O. The limiting position of ON is the tangent to Γ at O, which is also the polar of the parabola osculating C at O with respect to the point Q' on t such that $Q'O=OQ$. This limiting position of ON is the generalization of the affine normal.
T. R. Hollcroft (Aurora, N. Y.).

Tsuboko, Matsuji. On the evolutes of anharmonic curves in a projective plane. Proc. Phys.-Math. Soc. Japan (3) **23**, 571–578 (1941). [MF 5534]

An anharmonic curve is one which is projectively equivalent to its evolute and involute. Some properties of the projective normal, osculating conic and nodal cubic are derived. *M. S. Knebelman* (Pullman, Wash.).

Hsiung, Chuan-Chich. Sopra il contatto di due curve piane. Boll. Un. Mat. Ital. (2) **2**, 443–451 (1940). [MF 5563]

Bompiani [Atti Accad. Naz. Lincei. Rend. (6) **3**, 118–123 (1925)] showed the existence of a covariant line $r_0^{(k)}$ associated with the common point of two plane curves having a contact of order k at that point. The present paper gives geometrical characterizations of $r_0^{(k)}$ for $k=2, 3, 4$. A typical characterization may be quoted: Given two curves C and $\bar{C}$ having at O contact of the second order with t as their common tangent at O. Let C_2, $\bar{C}_2$ be the osculating conics of C and $\bar{C}$, respectively, at O, and C_3, $\bar{C}_3$ the cubics having with C and $\bar{C}$, respectively, five-point contact at O subject to the restriction that their intersections with t separate OI harmonically, where I is an arbitrary point on t. The polar line of t with respect to the lines joining O to the intersections of C_2, $\bar{C}_3$ and to the intersections of $\bar{C}_2$, C_3 is the line $r_0^{(2)}$. Similar characterizations are made for $r_0^{(3)}$, $r_0^{(4)}$. *V. G. Grove* (East Lansing, Mich.).

Bompiani, E. Sul contatto di due curve piane. Boll. Un. Mat. Ital. (2) **3**, 37–40 (1940). [MF 5575]

In the paper reviewed above a theorem is stated concerning a line projectively related to two curves having three-point contact at a given point. The present paper points out that the theorem is not true as stated; necessary restrictions are added to make it apply. *V. Snyder*.

Bompiani, E. Sistemi semplicemente infiniti di curve di una superficie aventi gli stessi piani osculatori. Boll. Un. Mat. Ital. (2) **3**, 97–101 (1940). [MF 5578]

Let S be a surface and Σ a system of ∞^2 planes. The problem is to determine the systems γ of ∞^1 curves Γ on S such that the planes of Σ are the osculating planes to the curves Γ. The author solves this problem by the following method. Let σ be the surface envelope of Σ. For every point P of σ let r be a straight line which is tangent to σ at P and also tangent to S. The systems γ on S are identical with the systems of the lines of regression of a system of developable surfaces of the congruence generated by the lines r. If S is a quadric, for every point S of σ we can find two lines r; and therefore on S there are two systems γ of curves Γ such that the system of their osculating planes is identical to the given system Σ. For this case the author carries out the necessary calculations. *G. Fubini* (Princeton, N. J.).

Bompiani, E. Geometrische Kennzeichnung der Flächen mit der Krümmung Null. Jber. Deutsch. Math. Verein. **51**, 82–100 (1941).

As is well known, the only two-dimensional surfaces of zero total curvature (plane developables) imbedded in three dimensional Euclidean space are the cylinder, cone and tangent surface to a space curve. These latter three surfaces are called "Torsen." The author's problem is to classify and characterize the real and complex two dimensional surfaces of zero total curvature which are imbedded in an n-dimensional Euclidean space. The classification is made by use of the dimensionality of the curvature space of the second order $S(2)$ (the curvature domain of the first order of Schouten-Struik). First, the case $S(2)=S_3$ (axial points) is discussed. It is shown that the only plane developables are the Torsen. Secondly, the case $S(2)=S_4$ (planar points) is investigated. A characterization of the plane developables in terms of the properties of a net of conjugate curves is given. Furthermore, if a non-isotropic surface possesses a family of asymptotic curves and is of zero total curvature, then these curves are shown to be a family of straight lines. This last conclusion is derived both synthetically and analytically. The author concludes this section by giving a synthetic method for determining these ruled surfaces which are not Torsen and which lie in a Euclidean space of four or more dimensions. Analytical examples of these surfaces are discussed. Finally, the last case $S(2)=S_5$ (spacial points) is examined. A characterization of this type of surface of total zero curvature is given.
N. Coburn (Austin, Tex.).

Löbell, Frank. Zur Differentialgeometrie der Regelscharen. Jber. Deutsch. Math. Verein. **51**, Abt. 2., 29–41 (1941).

Ruled surfaces are treated by the kinematical method. While one usually considers only one characteristic quantity, the parameter of distribution, the author introduces [following X. Antomari, Thèse, Paris, 1894] three differential invariants (derived from four relative invariants), which have an intuitive geometrical meaning and determine the ruled surface completely. Some applications (special surfaces) are given. *H. Samelson* (Princeton, N. J.).

Löbell, Frank. Die Bewegung des begleitenden Dreikants. Jber. Deutsch. Math. Verein. **51**, 148–150 (1941).

The kinematical method of the paper reviewed above is used to find a property which distinguishes the motion of the moving trihedral of a curve from the general motion of a rigid body. *H. Samelson* (Princeton, N. J.).

Srinivasiengar, C. N. The linear line-congruence. J. Indian Math. Soc. (N.S.) **5**, 73–91 (1941). [MF 5698]

The first part of the paper sets up a correspondence between the lines of a hyperbolic congruence and the points of a quadric in ordinary space. To a curve on the quadric there corresponds a ruled surface of the congruence. Properties of curves on a quadric may then be translated into theorems on the ruled surfaces of a linear congruence. The familiar theorem which states that through any point there pass three osculating planes of a twisted cubic, and the three points of osculation and the given point are coplanar translates into the theorem: through any ray of the linear congruence determined by the two directrices of a cubic scroll of the first species there pass three osculating quadrics. The quadric through the generators along which the osculation

takes place passes through the given ray. The theorems in this portion of the paper are projective theorems.

A second portion of the paper discusses metric properties of such congruences. Typical theorems may be stated as follows: Rays of a hyperbolic linear congruence for which the mean parameter of distribution is equal to a given constant lie on one or the other of two quartic scrolls having the directrices as double lines. The axes of the linear complexes of the asymptotic curves of any scroll of the hyperbolic congruence are all parallel to the middle plane, and their locus is a cylindroid. Of all scrolls belonging to a hyperbolic congruence, there are two families for which the Gaussian curvature at a given point P is a maximum or minimum. The asymptotic curve through P of a scroll of one family meets the ray again at a point Q such that the tangent plane at P to one of the scrolls of one family is also the tangent plane at Q to one of the other family. The two families having maximum or minimum Gaussian curvature at Q are identical with the two families of scrolls at P.

A final portion concerns itself with parabolic congruences. The following mode of generation of a parabolic linear congruence is given: Let a plane π meet a line l at O. Set up a (1, 1) correspondence between the lines through O on π and the points on l. Let OM on π correspond to the point P on l. Then the rays joining P to points of OM generate a parabolic linear congruence as P describes l. The rays of the congruence through the points on a curve on π generate a ruled surface of the congruence. In particular certain conics on π generate the cubic scroll of Cayley. *V. G. Grove.*

MacQueen, M. L. Osculating quadrics of ruled surfaces in reciprocal rectilinear congruences. Bull. Amer. Math. Soc. 47, 788–795 (1941). [MF 5490]

Let l_1 be the generator of a congruence Γ_1 and passing through the point x of an analytic surface S but not lying in the tangent plane to S at x. The author discusses the osculating quadrics of the two ruled surfaces generated by l_1 as x moves first along one and then the other of the asymptotic curves through x. A typical theorem may be stated as follows: The planes determined by l_1 and the asymptotic tangents intersect these osculating quadrics in l_1 and in lines which intersect l_1 in points which coincide if and only if the curves corresponding to the developables of Γ_1 or the reciprocal congruence Γ_2 form a conjugate net. The osculating quadrics intersect in the line l_1 and in a twisted cubic which intersects l_1 at the focal points of l_1. A similar discussion is made of similar quadrics associated with a line of the reciprocal congruence Γ_2.

V. G. Grove (East Lansing, Mich.).

Su, Buchin. Sopra certi fasci di quadriche e sul fascio canonico. Boll. Un. Mat. Ital. (2) 2, 438–443 (1940). [MF 5562]

In a previous note [Tôhoku Math. J. 39, 269–278 (1934)] the author introduced a one parameter family of Q_k of bundles of quadrics projectively associated with the fourth order neighborhoods of each point of a non-ruled surface in 3-space. With each such point there are also projectively associated two one parameter families of lines C_k^1, C_k^2, the canonical lines of the first and second kind. This note shows how the lines C_k^1, C_k^2 can be defined geometrically by a corresponding quadric bundle of the family Q_k.

J. L. Vanderslice (Bethlehem, Pa.).

Hsiung, Chuan-Chih. The canonical lines. Duke Math. J. 8, 738–742 (1941). [MF 5956]

Many of the lines covariantly related to a surface S at a point P pass through P and lie in a certain plane called the canonical plane. Any line lying in this plane and passing through P is called a canonical line of the first kind. The reciprocal polar of such a line with respect to the quadric of Lie of S at P is called a canonical line of the second kind. This paper concerns itself with new geometric characterizations for a general canonical line of each kind. A new characterization for the first axis of Čech is given. B. Su has constructed [Tôhoku Math. J. 39, 269–278 (1934)] a one parameter family of polarities C_h with respect to a quadric Q_h associated with S at P. A typical characterization may be stated as follows: Let V_i ($i=1, 2, 3$) be the poles of the osculating planes π_i of the curves of Darboux with respect to C_h. Then the harmonic polar of P with respect to the triangle $V_1V_2V_3$ is a canonical line of the second kind. By proper choice of the parameter h of C_h, this canonical line may be made to coincide with any canonical line of the second kind. A second characterization of these lines furnishes also a characterization of the canonical lines of the first kind. The polar line of the tangent plane at P to S with respect to the trihedron formed by the osculating planes π_i of the curves of Darboux is the first axis of Čech. *V. G. Grove* (East Lansing, Mich.).

Santaló, L. A. Quelques propriétés des courbes gauches dans la géométrie différentielle affine. Portugaliae Math. 3, 63–68 (1942). [MF 6234]

The author derives the differential equation of the developable surfaces through a curve in affine 3-space and discusses the case when the generators of the surface lie in a plane rigidly connected with the trihedral or are even themselves rigidly connected with it. *P. Scherk.*

Santaló, L. A. Curves of extremal total torsion and D-curves. Publ. Inst. Mat. Univ. Nac. Litoral 3, 131–156 (1941). (Spanish. English summary) [MF 5961]

From the author's summary: "D-curves are the curves on a surface at each of whose points the osculating sphere is tangent to the surface; these curves were studied by Darboux. In this paper I wish to establish that under certain conditions the D-curves coincide with the extremal curves of the total torsion, that is, with the curves for which the first variation of the integral $\int\tau ds$ is zero. In IV we apply these results to find the D-curves on the surface of the 'torus'; between these curves there are some closed curves which have positive torsion at each point." The proof is based upon the observation that the D-curves are characterized by the vanishing of the first variation of the total geodetical torsion along them. *P. Scherk.*

Maxia, A. Sui sistemi di curve tracciate su di una superficie. Boll. Un. Mat. Ital. (2) 3, 367–369 (1941). [MF 5703]

Let $x=x_1, x_2, x_3, x_4$ be homogeneous projective coordinates of a point on a non-developable surface S. Let us suppose that the parameter-lines u, v are not asymptotic lines of S. Let ρ be the intersection of the osculating planes to these lines and let K be the rectilinear congruence generated by the lines ρ. If X are the homogeneous coordinates of a point of ρ, we can find functions $a, b, c, \alpha, \beta, \gamma, p, q, r, s$ of u, v such that

$$x_{uu}=ax+bx_u+cX, \qquad x_{vv}=\alpha x+\beta x_v+\gamma X,$$
$$x_{uv}=px_u+qx_v+rx+sX.$$

From the conditions of integrability one can deduce two new equations:

$$(c\gamma-s^2)X_u=Ax_u+Bx_v+\lambda X+\mu x,$$
$$(c\gamma-s^2)X_v=Cx_u+Dx_v+\nu X+\sigma x.$$

The author remarks that $I=cC/\gamma B$ is a projective invariant (of third order) of the lines u, v and that

$$cdu^2+2sdudv+\gamma dv^2=0, \qquad Bdu^2+(D-A)dudv-Cdv^2=0$$

are the equations of the asymptotic lines of S and of the developable surfaces of K. The author gives also the geometrical interpretation of the equations

$$s(D-A)=0, \qquad \gamma B-cC=0, \qquad \gamma B-cC=s(D-A)$$

and remarks that one of these equations is a consequence of the other two. *G. Fubini* (Princeton, N. J.).

Urmajev, N. Die reduzierte Länge der geodätischen Linie. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 5, 369–376 (1941). (Russian. German summary) [MF 5841]

In geodesic polar coordinates on the ellipsoid of revolution, the arc length of a curve is given by $dS^2=ds^2+m^2d\alpha_1^2$, where s is the geodesic distance, α_1 the azimuth at the pole and m the reduced length given by $d^2m/ds^2+mK=0$, K being the Gaussian curvature. The author expands m in powers of s to terms of the seventh degree (he uses only fourth powers) and compares m for the ellipsoid with m_s for the sphere whose radius is the radius of mean curvature at the pole. If $v=(m-m_s)/m$ then $v_{\max}=\frac{1}{6}(s/N)^2e^2 \sin 2\phi$, where ϕ is the geocentric latitude and N is a function of the curvatures at the origin of polar coordinates.

M. S. Knebelman (Pullman, Wash.).

Ling, Donald and Recht, Leon. A theorem concerning the geodesics on a paraboloid of revolution. Bull. Amer. Math. Soc. 47, 934–937 (1941). [MF 5941]

The differential equations of the geodesics on a paraboloid of revolution can be integrated in terms of elementary functions and from these expressions the authors deduce the properties of the geodesics with regard to multiple points. Any one geodesic passes through a given point at most twice. Let a geodesic segment be denoted by $g(P, k)$ if it has the same initial and terminal point P and has k double points including P. Let $N(P, k)$ denote the maximum number of segments $g(P, k)$ corresponding to P and k. The authors prove that the paraboloid can be divided into a sequence of zones by planes perpendicular to the axis such that, if these are numbered in order $0, 1, 2, \cdots$, beginning with that containing the vertex, the following are true: (1) if P is an interior point of the kth zone, $N(P, j)=2$ if $j=1, 2, \cdots, k$; $N(P, j)=0$ if $j>k$; (2) if P is on that boundary of a zone which is nearer the vertex, $N(P, j)=2$ if $j=1, 2, \cdots, k-1$; $N(P, k)=1$; $N(P, j)=0$ if $j>k$.

G. A. Hedlund (Charlottesville, Va.).

Lieber, A. E. The first algebraic integrals of the equations of geodesy. C. R. (Doklady) Acad. Sci. URSS (N.S.) 31, 840–841 (1941). [MF 5512]

In order that $S_{a_1a_2\cdots a_p}$ should be the coefficients of a homogeneous first integral of the equations of the geodesics it is necessary and sufficient that $S_{(a_1a_2\cdots a_p,\beta)}=0$. By considering the integrability conditions of these equations it is shown that in the case of a surface there are at most $(p+1)(p+2)/2$ linearly independent first integrals.

M. S. Knebelman (Pullman, Wash.).

Reade, Maxwell and Beckenbach, E. F. Mean-value surfaces. Bull. Amer. Math. Soc. 47, 808–815 (1941). [MF 5493]

With a surface $S: x_j=x_j(u, v)$, $j=1, 2, 3$, the $x_j(u, v)$ being continuous in a domain D, the authors associate a family of mean value surfaces $S_\rho: x_j=A_{j\rho}(u, v)$, $j=1, 2, 3$, $A_{j\rho}(u, v)$ being the circular average of $x_j(u, v)$ over a circle of radius ρ. Adding differentiability conditions on the $x_j(u, v)$, the authors show that, if each surface S_ρ is given in isothermic representation by the above equation, then either (a) S is a spherical surface in representation whereby circles are mapped on circles, or (b) S is a minimal surface, and conversely. The nature of the mean value surfaces in each of the two cases (a), (b) is investigated. The work is based upon an earlier paper by these same authors [Trans. Amer. Math. Soc. 49, 354–377 (1941); these Rev. 2, 360].

W. T. Martin (Cambridge, Mass.).

Laura, Ernesto. Sull'equilibrio di una porzione di superficie conica flessibile, inestendibile, pesante. Boll. Un. Mat. Ital. (2) 2, 405–414 (1940). [MF 5557]

In this paper the author considers the problem of determining the form assumed by a portion of a conical, flexible and inextensible surface in a uniform gravitational field. The more general case in which the surface considered is a developable is treated in a paper to appear in the Atti dell'Istituto Veneto; it is there shown how the solution of this problem depends on the possibility of determining the Lagrangian parameters (multipliers), which appear in the equations of equilibrium due to Beltrami [Collected Works, v. III, p. 420], in terms of the characteristic elements of the surface. In the present note the author simplifies the analysis of the more special problem with which he is here concerned by referring the surface of equilibrium to a "moving trihedral" [Darboux] whose vertex describes the surface. It is shown that if the portion of conical surface is applicable to a circular sector, the problem can be solved by quadratures. [For a bibliography of the allied problem of the spherical catenary, see Appell, Mécanique Rationnelle, v. I.] *M. A. Basoco* (Lincoln, Neb.).

Bell, P. O. On differential geometry intrinsically connected with a surface element of projective arc length. Trans. Amer. Math. Soc. 50, 529–547 (1941). [MF 5722]

The paper under review concerns itself with a generalization of Fubini's element of projective arc length. The quadratic form defining this element is made to depend upon any whatever particular invariant R of a class of projective invariants. Among the members of this class is the invariant upon which is based Fubini's projective arc length. A definition of a projective non-Euclidean distance based upon a particular quadric of Darboux depending upon R is made. The integral of the element of arc length along a curve is exhibited as the limit of sums of these non-Euclidean distances. The method of Abramescu [C. R. Acad. Sci. Paris 209, 780–781 (1939); these Rev. 1, 170] for characterizing the tangents of Darboux was made to depend upon the Euclidean normal sections of the surface and the metric arc length of curves. Bell gives the following generalization of Abramescu's characterization: Let C_λ be a curve on the surface S at P. Let A_s, B_s be points on C_λ whose distances from P along C_λ are s and $-s$. The line joining A_s to B_s intersects the tangent plane to S at P in a point P_s. The limit of P_s as s tends to zero is a point P_0 distinct from P and lying on the tangent to C_λ at P. The

locus of P_0 as C_λ varies is a rational cubic curve. The lines joining P to the inflexion points of this cubic are the tangents of Darboux; moreover the line of inflexions is the R-harmonic line discovered by the reviewer [Bull. Amer. Math. Soc. **36**, 582–586 (1930)] and named by Bell.

Bompiani [Appendix II: I fondamenti geometrici della teoria proiettiva della curve e delle superficie, pp. 691–697 of vol. 1 of Fubini-Čech, Geometria Proiettiva Differenziale, Bologna, 1926] obtained geometric interpretations for certain invariants related to Fubini's asymptotic curvature κ_a, Bompiani's projective curvature κ and projective torsion T. These differential invariants are formed with respect to the projective arc length of Fubini. Bell generalizes these differential invariants by "forming them with respect to" his generalized projective arc length, obtaining in this manner the classes $\{\kappa_a\}$, $\{\kappa\}$, $\{T\}$, the members of each class depending upon the particular choice of R. Purely geometric interpretations are made for the classes $\{\kappa_a\}$, $\{\kappa\}$ and for a class $\{\bar{T}\}$ closely related to $\{T\}$. Two illustrative theorems may be stated as follows: A curve will be said to be a projective geodesic if it is an extremal curve of the invariant integral of the generalized projective element of arc length. Then a curve is a projective geodesic on a surface with respect to an invariant quadratic form if and only if its projective curvature with respect to the form vanishes. And a curve is a union curve of an R-conjugate congruence if and only if its projective curvature with respect to the form vanishes. Having previously [Amer. J. Math. **62**, 680–686 (1940); these Rev. **2**, 18] characterized the R-conjugate lines as the cusp axis with respect to the extremals of the invariant integrals associated with the generalized projective element of arc length, the author exhibits the Euclidean metric normal as the cusp axis of a certain invariant integral. *V. G. Grove* (East Lansing, Mich.).

Marcus, Ephr. Sur une correspondance remarquable entre droites et cubiques et une propriété d'une classe de surfaces isothermo-asymptotiques de Fubini. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 242–244 (1941). [MF 5860]

Let α be the tangent plane to a surface S at a point A. The author considers a special system Σ of ∞^2 cubic curves which lie on α and have a double point at A with tangents in the asymptotic directions of S. The other properties of the system Σ are not so simple and are less important. The line r joining the three points of inflection of a curve C of Σ is called the line of inflection r of C. The author studies the correspondence between the curves C and the straight lines r of the tangent plane α. Next the author considers the cubic C of Σ which has at A a contact of second order with the asymptotic lines of S and investigates the condition under which the corresponding line of inflection is a canonical line of second kind. He finds the corresponding surfaces S, and shows that they are isothermic-asymptotic. In this manner the author discovers a new class of isothermic-asymptotic surfaces. *G. Fubini* (Princeton, N. J.).

Radon, Johann. Über Tschebyscheff-Netze auf Drehflächen und eine Aufgabe der Variationsrechnung. Mitt. Math. Ges. Hamburg **8**, part 2, 147–151 (1940). [MF 4118]

The first part of this paper is devoted to the question of obtaining a coordinate system in which the metric of a surface of revolution takes the form

$$ds^2 = du^2 + dv^2 + 2\cos\omega du dv.$$

The transformation used can be regarded as mapping a zone of the surface of revolution onto a rectangle in the u, v plane. Then the problem is reversed. Two opposite sides of a prescribed rectangle in the u, v plane are required to map into circles of latitude with preassigned radii. It is found that the surface of revolution, in which these circles are as far as possible from one another, is a surface of constant negative curvature. *E. W. Titt* (Austin, Tex.).

Thomas, Heinz. Zur Frage des Gleichgewichts von Tschebyscheff-Netzen aus verknoteten und gespannten Fäden. Math. Z. **47**, 66–77 (1940). [MF 6298]

Consider on a surface S a Tschebyscheff net (u, v). The lines of this net divide the surface into infinitesimal parallelograms. We suppose that at every point of its boundary the surface S is subjected to forces acting in the directions of the curves u, v. Consequently all sides of the above parallelograms will be subjected to a certain tension. The author considers the problem of finding the nets such that the four forces acting on any vertex of the net are in equilibrium; he finds the corresponding conditions on the coefficients of the differential forms of S. He studies the special case when the two tensions acting on two consecutive sides of any parallelogram are equal to each other and investigates the problem of finding the corresponding nets by supposing that they are nets of rotation. Under this supposition he finds that the corresponding surface has a constant negative curvature and demonstrates the converse theorem also by supposing that the curvature of the surface is constant. *G. Fubini* (New York, N. Y.).

Kanitani, Jôyô. Généralisation des directrices de Wilczynski. Proc. Phys.-Math. Soc. Japan (3) **23**, 399–409 (1941). [MF 5404]

Let S be a surface in a three dimensional space with a projective connection. Then it is shown that it is possible to define two lines at each point of S which are analogous to the directrices of Wilczynski of a surface in a projective space. These are called the directrices of the first and second kind. A number of geometric properties of these directrices are obtained of which the following are examples: (1) The two directrices associated with the same point of S are reciprocal to each other with respect to S. (2) There exist two families of curves on S along each of which the directrices of the second kind generate a developable surface. *A. Fialkow* (Brooklyn, N. Y.).

Haantjes, J. Conformal differential geometry. Curves in conformal euclidean spaces. Nederl. Akad. Wetensch., Proc. **44**, 814–824 (1941). [MF 5918]

The object of this paper is to obtain intrinsic equations for a curve in Euclidean n-space which determine it up to conformal point transformations. The method is that of proportional Euclidean metrics rather than polyspherical coordinates and therefore ordinary tensor analysis is used. First of all a conformally invariant parameter and ennuple are associated with any given curve together with $n-2$ conformal curvature scalars. Then a covariant vector is constructed which, with the Christoffel symbols and conformal parameter, permits conformal differentiation along the curve. These elements with one additional scalar fit together into the author's set of Frenet equations, from which he proves that the $(n-1)$ scalars as functions of the conformal parameter are conformally intrinsic. *J. L. Vanderslice* (Bethlehem, Pa.).

Coleman, Robert, Jr. Conformal geometry of one-parameter families of curves. Bull. Amer. Math. Soc. 48, 94–102 (1942). [MF 6185]

Kasner [Conformal geometry, Proc. Fifth International Congress of Mathematicians, Cambridge, 1912, Cambridge University Press, 1913, pp. 81–87] began the development of the conformal geometry of sets of curves. The author obtains absolute differential conformal invariants of a family of curves $F: u(x, y)=c$ with invariant parameter c. Any such family may be considered as a Riemann surface with metric $ds^2=(u_x^2+u_y^2)(dx^2+dy^2)$. The family F possesses the invariants $\Delta=(u_{xx}+u_{yy})/(u_x^2+u_y^2)$ and the derivatives of Δ with respect to the Riemann arc lengths of the curves of F and their orthogonal trajectories. For a non-isothermal family, there are invariants independent of the parameter c, which are rational functions of Δ and its derivatives. The geodesic curvature of any curve of F is Δ, and the Gaussian curvature of F is $K=-\Delta_u-\Delta^2$. Finally isothermal families are generalized by making the Gaussian curvature of a family with arbitrary parameter constant. *J. De Cicco.*

Venturelli, Lucia. Sull'equilibrio di una porzione di ipersuperficie. Boll. Un. Mat. Ital. (2) 3, 375–383 (1941). [MF 5706]

The necessary conditions of Beltrami for the equilibrium of an ordinary two-dimensional surface are plausibly generalized so as to apply to an n-dimensional manifold imbedded in a Euclidean $(n+1)$-dimensional space. As the results are apparently without immediate physical meaning (except for the special case of Beltrami, $n=2$), the paper might be regarded as belonging to the field of pure geometry. *D. C. Lewis* (Durham, N. H.).

Lense, Josef. Über die Ableitungsgleichungen einer Mannigfaltigkeit im mehrdimensionalen, komplexen euklidischen Raum. Math. Z. 47, 78–84 (1940). [MF 6299]

Consider a manifold V_m of m dimensions in a Euclidean space of $n>m$ dimensions. Suppose that V_m is defined by means of analytic functions of m complex parameters $u_1, u_2, \cdots, u_m$. For every point of V_m we define a system of n linearly independent vectors $\eta_1, \eta_2, \cdots, \eta_n$. We write (1) $\eta_i\eta_j=B_{ij}$ and suppose (2) $\det|B_{ij}|\neq 0$. The derivatives of the η can be considered as linear combinations of the η. We will write

$$\partial\eta_\lambda/\partial u_\alpha=\sum_{\nu=1}^{n} C^{\nu}_{\lambda\alpha}\eta_\nu \tag{3}$$

so that

$$\partial B_{\lambda\mu}/\partial u_\alpha=\sum_{\nu=1}^{n}(C^{\nu}_{\lambda\alpha}B_{\nu\mu}+C^{\nu}_{\mu\alpha}B_{\nu\lambda}).$$

If the B and the C are known analytical functions of the u, the author states the conditions of integrability which will give a system of equations for the B and the C. If these conditions are satisfied, every solution of (3) will satisfy (1) identically if it satisfies (1) at one single point. If the first m vectors $\eta_1, \cdots, \eta_m$ are given by $\partial x/\partial u_\alpha=\eta_\alpha$ $(\alpha=1, 2, \cdots, m)$, the author finds that the C also satisfy the equations (4) $C_{\alpha\beta}=C_{\beta\alpha}$. If the C and the B are known functions satisfying the conditions of integrability, the equations (4) and the inequality (2), the manifold V_m (and the corresponding system of vectors η) will be determined up to a movement.

If the discriminant of the Gaussian linear element of V_m is of rank k, the author says that V_m is isotropic of rank k. The tangential space R_m of m dimensions is then also isotropic of rank k. The intersection of R_m with the normal space R_{n-m} of $n-m$ dimensions is a space R_{m-k} of $m-k$ dimensions, completely isotropic; the normal space R_{n-m} is isotropic of rank $n-2m+k$. In this case we can choose as vectors η the m vectors $\partial x/\partial u_\alpha$, $n-2m+k$ normal vectors n_r and other $m-k$ vectors subject only to the condition that the n vectors η be linearly independent. The author studies the case $k=0$, which he had studied previously [Math. Z. 34, 721–736 (1932)]. *G. Fubini.*

Ritter, Robert. Stationäre und extreme geometrische Ableitungen in Riemannschen Räumen. Anwendung auf die Flächenverbiegung. I. Jber. Deutsch. Math. Verein. 51, 101–124 (1941).

Let $\phi(x_1, x_2, \cdots, x_n)$ be an analytic point function in an n-dimensional Riemann space V_n. The author considers the variation of $d\phi/ds$ and $d^2\phi/ds^2$ (where s represents the arc length) evaluated along different curves C which pass through the same point P of V_n. He proves: (1) If the value of $d\phi/ds$ or of $d^2\phi/ds^2$ is stationary, then the direction of C is orthogonal to the hypersurface $\phi=$constant which contains P. (2) The stationary values of $d\phi/ds$ and of $d^2\phi/ds^2$ are $(\Delta_1\phi)^{\frac{1}{2}}$ and $(2\Delta_1\phi)^{-\frac{1}{2}}\cdot\Delta(\phi, \Delta_1\phi)$, respectively, where the Δ's are the differential parameters of first order. (3) The stationary value of $f(d\phi/ds, d^2\phi/ds^2)$ (where f is an analytic function) is equal to the value of f when the arguments $d\phi/ds$, $d^2\phi/ds^2$ assume their stationary values given by (2). There is also a discussion of the sufficient conditions that the stationary values of f be extreme values. The special cases (a) $n=2$, (b) C a geodesic of V_n are considered in detail. Various applications to surfaces are indicated. The method is straightforward. *A. Fialkow.*

Rosenfeld, B. Géométrie intérieure de l'ensemble des plans m-dimensionnels dans l'espace elliptique à n dimensions. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 5, 353–368 (1941). (Russian. French summary) [MF 5840]

The position of a straight line in three-space is given by six homogeneous coordinates which satisfy a quadratic relation; this may be interpreted in terms of a quadric hypersurface in elliptic five-space. In an earlier paper the author has studied this surface by methods of differential geometry [Bull. Acad. Sci. URSS. Sér. Math. 5, 105–126 (1941); cf. these Rev. 3, 17]. Here he studies in similar fashion the generalization in which m-dimensional flat spaces of an n-space take the place of straight lines; this leads to a variety of $m(n-m)$ dimensions obtained as the intersection of a number of quadric hypersurfaces. This variety is found to possess among others the following properties: its geodesics are related to what the author calls hyperhelicoids of the embedding space; it contains certain subvarieties with Euclidean metric which correspond to certain Abelian subgroups of the group of motions of the embedding space. *G. Y. Rainich* (Ann Arbor, Mich.).

Schapiro, J. L. Les réseaux de Tschebychef dans un espace Riemannien à n dimensions. C. R. (Doklady) Acad. Sci. URSS (N.S.) 32, 240–241 (1941). [MF 5859]

This paper deals with certain nets (n linearly independent fields) of curves in an n-dimensional Riemann space V_n each of which is called a Tschebychef net (réseau de Tschebychef). A holonomic net of curves is a Tschebychef net if any two hypersurfaces belonging to the same family of hypersurfaces (of the holonomic system) cut arcs of

equal lengths from all curves of the net which intersect them. The hypersurfaces are said to be contained by the net. The following results are typical: (1) If the first fundamental form of V_n, $ds^2=g_{ij}dx^idx^j$, is positive definite, the necessary and sufficient condition that the parametric curves form a Tschebychef net is that a coordinate system exist in which $g_{ii}=1$, $i=1, 2, \cdots, n$. (2) Under the same conditions, there exists a unique Tschebychef net which contains n arbitrary hypersurfaces which intersect at a point. (3) The necessary and sufficient condition that a net of curves be a Tschebychef net is that all the Poisson vectors formed from the unit tangent vectors of the net be zero.

A. Fialkow (Brooklyn, N. Y.).

Ghosh, N. N. A matrix treatment of rigid body motion in hyperspace. Bull. Calcutta Math. Soc. **32**, 109–120 (1940). [MF 6161]

Velocity and acceleration of points of a rigid body, the matrix of inertia, the moment of momentum and the system of forces acting on a rigid body in hyperspace are treated using a system of notations (previously introduced by the author) in which an n-vector is represented by an $(n+1)\times(n+1)$ matrix consisting of zeros except that the components of the vector appear in the first row and the first column. *G. Y. Rainich* (Ann Arbor, Mich.).

Ghosh, N. N. The tortuosity of a variety. Bull. Calcutta Math. Soc. **32**, 51–60 (1940). [MF 6156]

Given two sets of m vectors the author forms scalar products of each vector of the first set with each of the second and builds the determinant whose elements are these products; in earlier papers he used these "scalar determinants" in the process of orthogonalizing a system of vectors and in determining the angle between two linear subspaces of a hyperspace. In the present paper these results are applied to vectors and (osculating and normal) linear subspaces which arise as results of successive differentiations in connection with a curved variety immersed in a Euclidean hyperspace. *G. Y. Rainich.*

Coburn, N. Unitary spaces with corresponding geodesics. Bull. Amer. Math. Soc. **47**, 901–910 (1941). [MF 5935]

The geodesics of a unitary space K_n are found by a variation problem. The resulting Euler equation contains the torsion tensor. Then follow the necessary and sufficient conditions that two unitary spaces K_n, $'K_n$ with symmetric connections have corresponding geodesics, which condition is quite similar to the corresponding condition for Riemannian manifolds

$$'\Gamma^{\alpha}_{\nu\mu}=\Gamma^{\alpha}_{\nu\mu}+2p_{(\nu}A^{\alpha}_{\mu)}.$$

It is further shown that such a correspondence is impossible when K_n is with torsion, and $'K_n$ without it.

D. J. Struik (Cambridge, Mass.).

Schapiro, J. L. Sur les champs géodésiques de directions à plusieurs dimensions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 237–239 (1941). [MF 5858]

Let U^i be the components of a vector field in an n-dimensional space V_n having an affine connection. Then to each geodesic of V_n there corresponds a surface generated by the curves of the vector field which pass through this geodesic. The vector field U^i is called a geodesic vector field if the surface determined in this manner by each and every geodesic of the space is completely geodesic. The author determines necessary and sufficient conditions that an affine space admit a geodesic vector field. These conditions appear as a system of partial differential equations involving the covariant derivatives of U^i and the curvature tensor of V_n. If the coefficients of the affine connection of the space are the Christoffel symbols of the second kind computed from a positive definite Riemannian metric tensor, then the first fundamental form of V_n may be written (in an appropriate coordinate system) as

$$ds^2=(dx^n)^2+A(x^n)C_{ij}(x^k)dx^idx^j,\qquad i, j, k=1, 2, \cdots, n-1.$$

These results are generalized to holonomic vector fields of more than one dimension and to certain Riemann spaces whose first fundamental forms are indefinite.

A. Fialkow (Brooklyn, N. Y.).

Shabbar, Mohammad. Einstein spaces admitting the Lorentz group. Proc. Indian Acad. Sci., Sect. A. **14**, 133–138 (1941). [MF 5870]

This paper stems from work of D. D. Kosambi on Lorentz-invariant path equations [J. London Math. Soc. **15**, 86–91 (1940); cf. these Rev. **2**, 21]. The author starts from an explicit form (involving two arbitrary functions α, γ) of the most general Riemannian metric admitting the n-dimensional Lorentz group. For $n\geqq 4$ it is conformally flat. The conditions that it be an Einstein space reduce to one differential equation in α and γ which implies projective flatness and has explicit solutions in case Milne's condition is imposed on the paths. Finally, from the relativity field equations with stress-energy tensor, formulae are obtained for pressure and density in terms of α and γ.

J. L. Vanderslice (Bethlehem, Pa.).

Fialkow, Aaron. Correction to "Totally geodesic Einstein spaces." Bull. Amer. Math. Soc. **48**, 167–168 (1942). [MF 6198]

The paper appeared in the same Bull. **45**, 423–428 (1939).

Ostrowski, Alexander. On the definition of contact transformations. Bull. Amer. Math. Soc. **47**, 760–763 (1941). [MF 5483]

If $z=f(x_1, \cdots, x_n)$ and $\partial z/\partial x_\nu=p_\nu$, the transformation (1) $Z=Z(z, x, p)$, $X_\nu=X_\nu(z, x, p)$ is reversible if (1) and $P_\nu=\partial Z/\partial X_\nu$ are solvable uniquely for $z=z(Z, X, P)$ and $x_\nu=x_\nu(Z, X, P)$. The author shows that in the case of hypersurfaces a reversible transformation is necessarily a contact transformation (of the first order). The author also shows by an example that in the case of a curve (in a space of more than two dimensions) there are reversible transformations which are not extensions of point transformations and are therefore not contact transformations.

M. S. Knebelman (Pullman, Wash.).

Kowalewski, Gerhard. Zur natürlichen Geometrie der irreduziblen G_8 von Berührungstransformationen. J. Reine Angew. Math. **183**, 243–250 (1941). [MF 5726]

The irreducible G_8 of contact transformations: p, q, $xp-y_1q_1$, $xq+q_1$, x^2q+2xq_1, $2y_1p+y_1^2q$ [Lie-Engels, Transformationsgruppen, Leipzig, 1890, vol. 2], may be obtained from the unimodular affine transformation $\bar{x}=a_1x+a_2z+a$, $\bar{z}=b_1x+b_2z+b$, $a_1b_2-a_2b_1=1$ by defining $y=\int z\,dx$. If $u_k=d^ku/dx^k$, it is shown that $\bar{z}_2=z_2/(a_1+a_2z_1)^3$ from which it follows that $ds=z_2^{1/3}dx$ is invariant and is taken as arc length. The next invariant obtained is $J=d(y_3^{-5/3}y_4)/dx$. The J-curves of the group are defined as curves along which J is a constant. If $J=0$ the curve is a singular J curve. By

choosing the initial element to be $(0,0,0,0,1,0)$, it is shown that the singular J-curve is the cubical parabola $y=x^3/6$ and the non-singular J-curves through this element are

$$y=\frac{s}{2k^2}+\frac{\sinh 2ks}{4k^3}-\frac{\sinh ks}{k^3},\quad x=\frac{\sinh ks}{k}\quad \text{if}\quad J=-3k^2,$$

$$y=-\frac{s}{2k^2}-\frac{\sin 2ks}{4k^3}+\frac{\sin ks}{k^3},\quad x=\frac{\sin ks}{k}\quad \text{if}\quad J=3k^2.$$

The corresponding s curves in the above three cases are a parabola, a hyperbola and an ellipse. By considering the extremals of $\int y_3^{1/3}dx$ it is shown that the J-curves are the geodesics of the group. *M. S. Knebelman.*

Takasu, Tsurusaburo. Gemeinsame Behandlung der Differential- und Integralinvarianten der Kurven und der "Horn Angles" in den elliptischen konformen, parabolischen konformen und hyperbolischen konformen Ebenen. Proc. Imp. Acad. Tokyo 17, 163–169 (1941). [MF 5395]

Takasu, Tsurusaburo. Gemeinsame Behandlung der Äquivalenzprobleme der Kurven in der elliptischen konformen, parabolischen konformen und hyperbolischen konformen Ebene. Proc. Imp. Acad. Tokyo 17, 170–174 (1941). [MF 5396]

Takasu, Tsurusaburo. Gemeinsame Behandlung der Äquivalenzprobleme der Kurven in der elliptischen Laguerreschen, parabolischen Laguerreschen und hyperbolischen Laguerreschen Ebene. Proc. Imp. Acad. Tokyo 17, 175–177 (1941). [MF 5397]

Further exercises in the author's previously announced elementary geometries [Proc. Imp. Acad. Tokyo 16, 333–349 (1940); these Rev. 2, 166]. The Schwarz derivative and its generalizations play a prominent role in the calculations of the first two notes. *A. D. Michal.*

Yano, Kentaro et Mutô, Yosio. Sur la théorie des espaces à connexion conforme normale et la géométrie conforme des espaces de Riemann. J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. 4, 117–169 (1941). [MF 5633]

This paper first of all sets forth in some detail investigations on general conformal spaces previously published by the authors: the introduction and normalization of a conformal connection following the methods of Cartan and Veblen, the conformal theory of curves and general subspaces based on this normal connection. Then, as an application, the Frenet and Gauss-Codazzi equations for a conformal Riemann space are derived directly from the corresponding equations of the general theory. Indeed throughout the paper stress is laid on relationships with conformal Riemann space. A necessary and sufficient condition is given that the connection induced in a subspace be normal. A chapter is devoted to curves on subspaces, particularly generalized circles in relation to umbilical and quasi-umbilical subspaces. There is a new definition of autoconcurrent curves. An investigation of two-dimensional subspaces leads in particular to conditions for existence of totally umbilical 2-spaces and a geometric interpretation of generalized circles. *J. L. Vanderslice.*

Bortolotti, Enea. Duale Verwandtschaften, anholonome Flächen im projektiven und im affinen Raume. Jber. Deutsch. Math. Verein. 51, 151–169 (1941).

The equation $dz=pdx+qdy$, where p and q are functions of x, y, z, defines an anholonomic surface X_3^2 in space of three dimensions. A survey is given of results reached in the projective and affine theory of these "surfaces," mainly those dealing with properties of the second, third and fourth orders, and with dual relationships. There are observations concerning X_n^m, $m>2$, $n>3$. An excellent bibliography is attached. *D. J. Struik* (Cambridge, Mass.).

Bortolotti, Enea. Contributi alla teoria delle connessioni. II. Connessioni di specie superiore, fondamenti analitici calcolo del Vitali generalizzato. Mem. Ist. Lombardo. Cl. Sci. Mat. Nat. 24, 1–39 (1939). [MF 6020]

This paper can be considered as an amplification of the exposition of Vitali's absolute calculus [presented by the author in Atti Accad. Naz. Lincei. Rend. (6) 19, 777–781, 854–859 (1934)]. It is, at the same time, the second part of a series of papers of which part I appeared in Ann. Mat. Pura Appl. (4) 15, 1–45, 129–154 (1936), with co-author V. Hlavaty. Starting with reciprocal matrices, superdetermined Cartesian reference systems in an affine space E_n (sistema cartesiano sovrabbondanta) and affine spaces E_N associated with points of a manifold X_n^m with the related affine connection of König, the foundation is laid for the generalized calculus of Vitali. This consists in associating with a point of X_n^m an infinite number of affine spaces $E_{(1)}, E_{(2)}, \cdots, E_{(\nu)}, \cdots$, each space contained in the next space, with dimensions $N_1, N_2, \cdots, N_\nu, \cdots$, respectively $(N_1\leqq N_2\leqq\cdots\leqq N_\nu\leqq\cdots)$. Let there be a set of numbers $M_1, M_2, \cdots, M_\nu, \cdots$, taken from a "campo di aggrupamenti di numeri" T_ν, for which $M_\nu\geqq N_\nu$, $\nu=1,2,3,\cdots$. In every $E_{(\nu)}$ a Cartesian reference system $e_{\nu(A_\nu)}$ is selected with M_ν vectors, the A_ν being the group of numbers taken from T_ν, the reference vectors of $E_{(\nu)}$ being part of the reference vectors of $E_{(\nu+1)}$. A transformation of these vectors in $E_{(\nu)}$, which leaves this property invariant, is

$$e_{\nu(A_\nu')}=U_{A_\nu'}^{A_\nu}e_{\nu(A_\nu)}.$$

By means of this system linear connections of the König type can be introduced, with corresponding Γ. The condition of Vitali follows, expressing the fact that the law of construction of the U be independent of the choice of Γ. The calculus of Vitali is now built upon this structure, with super and subscripts denoting series of indices. The definition is extended to tensors of noninteger classes, for which also covariant derivatives can be defined. For the many details and applications we must refer to the original paper, which has also an excellent bibliography. *D. J. Struik.*

Kosambi, D. D. Path equations admitting the Lorentz group. II. J. Indian Math. Soc. (N.S.) 5, 62–72 (1941). [MF 5697]

[The first part appeared in J. London Math. Soc. 15, 86–91 (1940); cf. these Rev. 2, 21.] According to the author this paper, by giving rather elementary results in the theory of Lorentz invariant path spaces, unifies and illustrates work of his published elsewhere. Assuming the form of the path equations found in Part I, questions are asked regarding the existence of projective flatness, isotropy (in the sense of Kosambi), Riemann or Finsler metric, together with their interrelations. Various sets of differential equations result and the existence and general form of solution are investigated. Specific results are too diverse to be given here. *J. L. Vanderslice* (Bethlehem, Pa.).